Consensus of Multiagent Systems With Distance-Dependent Communication Networks

Gangshan Jing, Yuanshi Zheng, and Long Wang

***Abstract*— In this paper, we study the consensus problem of discrete-time and continuous-time multiagent systems with distance-dependent communication networks, respectively. The communication weight between any two agents is assumed to be a nonincreasing function of their distance. First, we consider the networks with fixed connectivity. In this case, the interaction between adjacent agents always exists but the influence could possibly become negligible if the distance is long enough. We show that consensus can be reached under arbitrary initial states if the decay rate of the communication weight is less than a given bound. Second, we study the networks with distance-dependent connectivity. It is assumed that any two agents interact with each other if and only if their distance does not exceed a fixed range. With the validity of some conditions related to the property of the initial communication graph, we prove that consensus can be achieved asymptotically. Third, we present some applications of the main results to opinion consensus problems and formation control problems. Finally, several simulation examples are presented to illustrate the effectiveness of the theoretical findings.**

***Index Terms*— Formation control, multiagent systems, opinion dynamics, rendezvous, state-dependent graph, switching networks.**

I. Introduction

A. Motivation

DISTRIBUTED coordination of systems with multiple agents has attracted much attention from different research communities in recent years. In these systems, agents interact with each other via a communication topology and only employ local information. As a result, in order to drive them to accomplish tasks, a distributed control law is required. Multiagent systems have a wide range of applications, since they can perform a variety of collective behaviors, including formation of unmanned aerial vehicles [1]–[5], attitude adjustment of spacecrafts [6], and flocking of multiple robots [7], [8], to name but a few. As a typical paradigm of these challenging topics, the consensus problem has been studied for a long time [9], [10]. Consensus is said to be reached if a group of agents agree upon a certain quantity of interest depending on their states. In the literature, the consistent value might represent physical quantities, such as position [11], [12], heading angle [13], opinion [14]–[19], or temperature [20].

B. Related Work

Due to unreliable communication, limited sensing range, or heterogeneity of different agents, consensus problems are often considered under a switching communication graph. To date, there have been numerous results related to this issue [21]–[32]. However, all these results are concerned under an assumption that the switching communication graph is time-dependent. In reality, multiagent systems are often subject to distance-dependent communication networks. For instance, in formation control of mobile robots, every robot can only interact with others within its sensing range due to the limitation of visibility [12]. Another example is Vicsek's swarming model [33] established for the observed collective behavior in nature. In the model, all agents keep the same speed but different headings. Every agent updates its heading by averaging headings of the agents who stay close to it. In such circumstances, the communication graph is dependent on agents' relative states. The information transmission weight between the adjacent agents varies with their distance, which leads to the fact that the connectivity of the communication network can be possibly broken due to the motion of agents. As a result, consensus cannot be guaranteed. Although distance-dependent communication graphs can be viewed as a case of time-varying communication graphs studied in [29], we believe that the consensus model under distance-dependent interactions has its specific properties and remains to be further studied.

A few experimental and theoretical efforts have been carried out on coordination control with distance-dependent communication networks. In [34], the Cucker–Smale (C-S) flocking model via a transmission weight depending on the Euclidean distance between agents is investigated. The weight is designed like gravity, i.e., as the distance between any two agents increases, the information that they receive from each other weakens but always exists. That is, the communication graph is always complete. The authors' research shows that convergence can be achieved under a condition on the initial states, which is really different from the previous results of systems in time-dependent switching networks. A model of opinion dynamics is introduced in [15]. It describes the evolution of a number of opinions belonging to a group of interacting agents. An agent interacts with another agent if their opinions differ by less than a fixed value. Several results related to opinion dynamics have been conducted to show the convergence [14], [16], [17]. Similarly, the rendezvous problem of multiagent

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systems also involves the uncertainty of the network connectivity [11], [12], [35]. In their works, the protocol moves every agent toward a computed target point at each time instant. By limiting the allowable motion of each agent, the network connectivity is maintained. In addition, the information transmitted between the adjacent agents in [36] and [37] is influenced by their state difference. However, all the aforementioned investigations are discussed without giving consideration to connectivity maintenance under a simple consensus algorithm. Yang *et al.* [18] presented a sufficient condition for consensus of the continuous-time opinion formation model by limiting the number of common neighbors shared by any two agents. Nevertheless, this condition remains to be relaxed if the distribution of the opinions at the initial time is uneven. All the above investigations are related to multiple agents with single-integrator dynamics. In problems of flocking, rendezvous, and formation control, the control input of a robot is often considered as its acceleration rather than speed. As a result, it is also significant to study the consensus problems for double-integrator agents under the distance-dependent interactions. In engineering applications, due to the limited bandwidth of networks, the control input of a continuous-time system may use the data sampled at discrete time instants. In such circumstances, the system should be described as a discrete-time model. This motivates us to investigate discrete-time multiagent systems with the second-order dynamics.

C. Our Contributions

Out of the above-mentioned situation, this paper studies the consensus problem of multiagent systems with a general distance-dependent communication weight. Both discrete-time and continuous-time dynamics are considered. For each case, two types of distance-dependent communications are investigated. In the first networks, we fix the connectivity of the communication topology, whereas the communication weight between the adjacent agents always varies with their distance. Such a communication network includes the one considered in the C-S flocking model as a special case. By constructing novel Lyapunov functions, it is found that under a connected communication graph, consensus can be globally reached except for continuous-time multiagent systems with double-integrator dynamics applying the dynamic consensus protocol. This is inconsistent with the intuition that preserving connectivity is sufficient for reaching consensus. In the above special case, the communication weight should satisfy some specified condition for the realization of consensus. In the second type of networks, we assume that the communication graph is fully dependent on the interdistance of the agents. That is, the network connectivity is also distance-dependent and may be broken as the system evolves. Different from [14] and [16], the conditions presented in this paper can make all agents rather than the agents in each connected component asymptotically reach consensus. Compared with the results in [18], our conditions do not limit the number of common neighbors shared by any pair of agents and are more relaxed if there are large differences between several opinions at the initial time.

In this paper, we assume that the interaction between adjacent agents decays as their distance increases. This assumption can be removed in several circumstances. Our results are applied to solving opinion consensus problems and formation control problems.

Notation: Throughout this paper, we denote the set of real numbers by $\mathbb{R}$, the set of positive real numbers by $\mathbb{R}_{>0}$, and the set of nonnegative real numbers by $\mathbb{R}_{\geq 0}$. Let $\mathbb{R}^n$ be the $n-$dimensional Euclidean space and $\|\cdot\|$ be the Euclidean norm. X^T stands for the transpose of matrix X and $|\mathcal{V}|$ is the cardinality of set $\mathcal{V}$. $\pi_M(x)$ denotes the orthogonal projection of x onto space M. $\dim(M)$ is the dimension of space M. $\otimes$ represents the Kronecker product. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, $\lambda_i(A)$ denotes the ith eigenvalue of A, i.e., $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$. $H_0(A)$ denotes the set of all eigenvectors of matrix A corresponding to zero eigenvalue. $\{i, j\}$ denotes a vertex set, including the ith vertex and the jth vertex. $\lfloor x \rfloor$ is the largest integer not greater than x and $\lceil x \rceil$ is the smallest integer not less than x.

II. Problem Formulation

A. Preliminaries of Graph Theory

We use a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ to denote the communication relationship between the agents. $\mathcal{V} = \{1, \ldots, n\}$ is a set denoting n agents in the system. $\mathcal{E}$ is the set of edges, and each edge is denoted by a pair of agents, i.e., (i, j). In this paper, we propose a matrix $G = [g_{ij}] \in \mathbb{R}^{n\times n}$ to show the distribution of communication links in the network. That is, $g_{ij} = 1$ if $(j, i) \in \mathcal{E}$, and $g_{ij} = 0$ otherwise. The set of neighbors of Agent i is denoted by $\mathcal{N}_i = \{j \mid g_{ij} = 1, j \in \mathcal{V}\}$. $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is a matrix describing the weight of information flow between agents, in which a_{ij} denotes the information transmission weight between agents i and j. A graph $\mathcal{G}$ is undirected if G and $\mathcal{A}$ are both symmetric matrices. We use a diagonal matrix $\Delta = [\Delta_{ij}]$ with $\Delta_{ii} = \sum_{j\in\mathcal{V}} a_{ij}$ to show the degree of each agent, and the Laplacian matrix of graph $\mathcal{G}$ is defined by $L = \Delta - \mathcal{A}$. By Gerschgorin's theorem, it can be easily proved that L is a positive semi-definite matrix. In this paper, the communications between the agents may be always changing as agents' states evolve. Hence, we use L_x to denote the Laplacian matrix according to state x for continuous-time multiagent systems, and L_t to denote the Laplacian matrix at step t for discrete-time multi-agent systems. A path from i to j in graph $\mathcal{G}$ is a sequence of distinct edges of the form $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$, where $i_1 = i$, $i_k = j$, and $(i_r, i_{r+1}) \in \mathcal{E}$ for $r \in \{1, \ldots, k-1\}$. A graph is said to contain a directed spanning tree if it is directed, and there exists a vertex called the root, such that every other vertex in the graph is connected with the root by a path starting from the root. A graph is said to be connected if it is undirected, and there exists a path between any two distinct vertices of the graph. Two paths are said to be disjoint if they have no common edges. Throughout this paper, we always assume that graph $\mathcal{G}$ is undirected unless otherwise specified.

The connectivity of graph $\mathcal{G}$, written by $\kappa(\mathcal{G})$, is the minimum size of a vertex set S, such that $\mathcal{G}' = (\mathcal{V} - S, \mathcal{E}', \mathcal{A}')$ is disconnected or has only one vertex, where $\mathcal{E}'$ and $\mathcal{A}'$ are the corresponding edge set and the adjacency matrix when all the vertices in S are deleted from $\mathcal{G}$. Therefore, $\kappa(\mathcal{G})$ can be confirmed only by G. Furthermore, it is straightforward to see that $\kappa(\mathcal{G}) > 0$ if and only if $\mathcal{G}$ is connected. Given $i, j \in \mathcal{V}(\mathcal{G})$, a set $S \subseteq \mathcal{V}(\mathcal{G}) - \{i, j\}$ is an $i, j-$cut if after deleting all vertices of S from $\mathcal{G}$, no path between i and j exists in the graph.

B. System Models

For discrete-time multiagent systems, we consider the agents with both the first-order dynamics

$$x_i(t+1) = x_i(t) + u_i(t), \quad i \in \mathcal{V} \tag{1}$$

and the second-order dynamics

$$\begin{aligned} x_i(t+1) &= x_i(t) + k_1 v_i(t) \\ v_i(t+1) &= v_i(t) + u_i(t), \quad i \in \mathcal{V} \end{aligned} \tag{2}$$

where $k_1 > 0$ is the control gain.

For continuous-time multiagent systems, agents with both single-integrator dynamics

$$\dot{x}_i = u_i, \quad i \in \mathcal{V} \tag{3}$$

and double-integrator dynamics

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= u_i, \quad i \in \mathcal{V} \end{aligned} \tag{4}$$

are considered.

In the above systems, $x_i \in \mathbb{R}^m$ and $v_i \in \mathbb{R}^m$ denote the position-like state and the velocity-like state of Agent i, respectively, and $u_i \in \mathbb{R}^m$ is the control input applying to Agent i, where m is a positive integer. In this paper, $\|x_i - x_j\|$ is considered as the distance between Agents i and j. Let $\mathbb{E} = \mathbb{R}^m$, then $x = (x_1^T, \ldots, x_n^T)^T,\ v = (v_1^T, \ldots, v_n^T)^T \in \mathbb{E}^n$. In the following, a matrix in $\mathbb{R}^{n\times n}$ may act on $\mathbb{E}^n$. That is, $Ax = (A \otimes I_m)x$ for $A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{E}^n$. We say that the consensus problem is solved if x converges into $M = \text{span}\{\mathbf{1}_n \otimes r \mid r \in \mathbb{E}\}$ as $t \to \infty$. In particular, if $M = \{\mathbf{1}_n \otimes ((1)/(n)) \sum_{i\in\mathcal{V}} x_i(0)\}$, the average consensus is said to be solved. Let e_i denote the canonical vector with a 1 in the ith entry and 0's elsewhere. Note that e_i, $i = 1, \ldots, m$ are the standard orthogonal bases of $\mathbb{R}^m$. Then, $f_i = ((1)/(\sqrt{n}))\mathbf{1}_n \otimes e_i$, $i = 1, \ldots, m$ are the standard orthogonal bases of M. Therefore, the orthogonal projection of x onto M is

$$\begin{aligned} \pi_M(x) &= \sum_{i=1}^{m} \langle x, f_i\rangle f_i \\ &= \sum_{i=1}^{m} \left\langle x, \frac{1}{\sqrt{n}}\mathbf{1}_n \otimes e_i \right\rangle \cdot \frac{1}{\sqrt{n}}\mathbf{1}_n \otimes e_i \\ &= \mathbf{1}_n \otimes \frac{1}{n}\sum_{i\in\mathcal{V}} x_i. \end{aligned}$$

We set $p = x - \pi_M(x)$ and $q = v - \pi_M(v)$, then consensus is reached if and only if $\|p\| \to 0$ and $\|q\| \to 0$ as $t \to \infty$.

C. Useful Lemmas

Several lemmas associated with graphs and matrices are listed as follows.

Lemma 1 [26]: If the communication graph $\mathcal{G}$ is connected, then $H_0(L \otimes I_m) = \text{span}\{\mathbf{1}_n \otimes r \mid r \in \mathbb{E}\} = M$, where L is the Laplacian matrix of $\mathcal{G}$.

Lemma 2 [38]: Given a positive semidefinite $d \times d$ matrix A with a simple zero eigenvalue, we have $x^T Ax \geq \lambda_2(A)\|x - \pi_{H_0(A)}(x)\|^2$, for any $x \in \mathbb{R}^d$.

Lemma 3 [34]: For all $x \in \mathbb{E}^n$, $L \in \mathbb{R}^n$ is the Laplacian matrix of a graph, we have the following.

1) $\|x_i - x_j\| = \|p_i - p_j\| \leq \sqrt{2}\|p\|$.
2) $((1)/(2n)) \sum_{i\in\mathcal{V}} \sum_{j\in\mathcal{V}} \|x_i - x_j\|^2 = \|p\|^2$.
3) $\langle x, Lx\rangle = ((1)/(2)) \sum_{i\in\mathcal{V}} \sum_{j\in\mathcal{V}} a_{ij}(x)\|x_i - x_j\|^2 \geq 0$.

Lemma 4: Suppose that the connectivity of graph $\mathcal{G}$ is $\kappa(\mathcal{G}) = k^* > 0$, then there exist at least k^* disjoint paths between any different vertices.

Proof: See the Appendix. ■

Lemma 5: If there are less than $n - 1$ pairs of disconnected vertices in graph $\mathcal{G}$, then $\mathcal{G}$ is connected.

Proof: See the Appendix. ■

III. Consensus of Discrete-Time Multiagent Systems

We study the consensus problem of discrete-time multiagent systems in this section. By using Lyapunov methods, several consensus criteria are obtained.

A. Distance-Dependent Communication Weight

We consider two classes of communication networks. The first case is of fixed connectivity, which implies that G and $\kappa(\mathcal{G})$ are invariant. The communication weight between Agents i and j is set by $a_{ij} = g_{ij}\alpha(\|x_i - x_j\|^2)$, where $\alpha(s)$ is a positive function, which decays as s increases. For Agent i, the information that it receives from Agent j can be denoted by $g_{ij}\alpha(\|x_i - x_j\|^2)(x_i - x_j)$. Assumption 1 is made for $\alpha(\cdot)$.

Assumption 1: $\alpha(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is nonincreasing, $\alpha(0) < \infty$.

In the second case, the distribution of agents' states totally determines the communication graph and in turn determines the connectivity of the network. More specifically, the communication weight between i and j is $a_{ij} = g_{ij}\alpha(\|x_i - x_j\|^2) = \alpha(\|x_i - x_j\|^2)$, because $g_{ij} = 1$ if and only if $\alpha(\|x_i - x_j\|^2) \neq 0$. Here, $\alpha(\cdot)$ satisfies Assumption 2.

Assumption 2: $\alpha(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is nonincreasing, $0 < \alpha(0) < \infty$, $\alpha(s) \geq \alpha^*$ for some constant $\alpha^* > 0$ if $s < R^2$, $\alpha(s) = 0$ if $s \geq R^2$, where $R \in \mathbb{R}_{>0}$ is a constant.

For the sake of simplicity, we denote $\alpha(\|x_i - x_j\|^2)$ by $\alpha_{ij}(x)$ in the rest of this paper.

B. Lyapunov-Like Function

Now, we establish a function $w(z) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ for constructing Lyapunov functions in the following:

$$w(z) = \begin{cases} \alpha(r)z, & 0 \leq z < r \\ \sum_{s=1}^{\lfloor \frac{z}{r} \rfloor} \alpha(sr)r + \alpha\left(\left\lceil \frac{z}{r} \right\rceil r\right)\left(z - \left\lfloor \frac{z}{r} \right\rfloor r\right), & z \geq r \end{cases}$$

where $\alpha(\cdot)$ is a nonincreasing function and r is a positive constant.

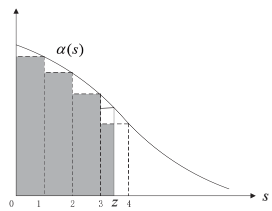


Fig. 1. $w(z)$ with $r = 1$ and $z = 3.5$.

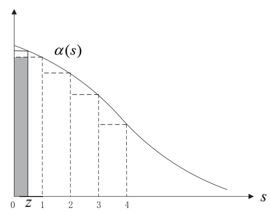


Fig. 2. $w(z)$ with $r = 1$ and $z = 0.5$.

For better understanding $w(z)$, we present an example to express the relationship between $w(\cdot)$ and $\alpha(\cdot)$. Let $r = 1$, the area of the shaded part in Fig. 1 is equal to $w(3.5)$, and the one in Fig. 2 is equal to $w(0.5)$. For simplicity, we define $x_{ij}(t) = x_i(t) - x_j(t)$, $W_{ij}(t) = w(\|x_{ij}(t)\|^2)$, and $W(t) = (1/2)\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}} g_{ij}W_{ij}(t)$. Proposition 1 shows some properties of W.

Proposition 1: For any $z \geq 0$, the following statements hold.

1) Suppose that the communication graph $\mathcal{G}$ contains a directed spanning tree. Then, $W(t) = 0$ if and only if $x_i = x_j$ for any $i, j \in \mathcal{V}$.
2) For a fixed r, $w(z)$ is increasing of z.
3) For all $t \geq 0$
$$W(t+1) - W(t) \leq \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}} g_{ij}\alpha_{ij}(x(t)) \times(\|x_{ij}(t+1)\|^2 - \|x_{ij}(t)\|^2). \quad (5)$$
4) $\lim_{r\to 0} w(z) = \int_0^z \alpha(s)ds$ for $0 \leq z < \infty$.
5) If $a_{ij} = \alpha(\|x_i - x_j\|^2)$ satisfies Assumption 2, and $W(t) < (n-1)w(R^2)$ for some $r \in (0, R^2)$, then graph $\mathcal{G}(t)$ is connected.

Proof: See the Appendix. ■

Remark 1: We can see that $w(z)$ is the approximation of $\int_0^z \alpha(s)ds$ in some sense. With the decreasing of r, $w(z)$ is closer to $\int_0^z \alpha(s)ds$. Actually, when we let $w(z) = \int_0^z \alpha(s)ds$, 1)–3) in Proposition 1 also hold. The corresponding proof is similar. In the rest of this paper, we define $w(z) = \int_0^z \alpha(s)ds$ for $r = 0$.

C. Consensus in Networks With Fixed Connectivity

When a_{ij} satisfies Assumption 1, a long distance between a pair of agents may cause their interactions to be slight and not effective. For reaching consensus, we hope to obtain a bound of the distance between any agents. In the results, we will see that the boundedness of $\|p\|$ is the key to solve the consensus problem. Once $\|p\|$ is guaranteed to be bounded, Lemma 6 shows that the algebraic connectivity of the communication graph, written by $\lambda_2(L)$, has a positive lower bound.

Lemma 6: Under Assumption 1, if $\|p(t)\|$ is bounded, and the communication graph is connected. Then, $\lambda_2(L_t)$ has a positive lower bound.

Proof: See the Appendix. ■

For agents with dynamics (1), the consensus protocol is given by
$$u_i(t) = h\sum_{j=1}^{n} g_{ij}\alpha_{ij}(x(t))(x_j(t) - x_i(t)) \quad (6)$$
where $h > 0$ is the control gain, $i \in \mathcal{V}$.

Theorem 1: Consider a system consisting of n agents with dynamics (1). Under Assumption 1, protocol (6) globally asymptotically solves the average consensus problem if the communication topology is connected and $h < (1/(d_{\max}\alpha(0)))$, where $d_{\max}$ is the maximum degree among all agents.

Proof: The discrete-time multiagent system (1) with protocol (6) can be written as
$$p(t+1) = p(t) - hL_t p(t).$$
Consider $V(t) = \|p(t)\|^2$ as a Lyapunov function. It holds that
$$\begin{aligned} V(t+1) - V(t) &= p^T(t)(I - hL_t)^2 p(t) - p^T(t)p(t) \\ &= p^T\left(-2hL_t + h^2L_t^2\right)p \\ &= -p^T\Xi_t p. \end{aligned}$$
The eigenvalues of Ξ_t are denoted by $\xi_i = 2h\lambda_i(L_t) - h^2\lambda_i^2(L_t) = h\lambda_i(L_t)(2 - h\lambda_i(L_t))$. From Gerschgorin's theorem, $\lambda_i(L_t) \leq \max_{i\in\mathcal{V}}\{2\sum_{j\in\mathcal{N}_i}\alpha_{ij}(x)\} \leq 2d_{\max}\alpha(0)$. Then, $h\lambda_i(L_t) - 2 < (1/(d_{\max}\alpha(0)))\cdot 2d_{\max}\alpha(0) - 2 = 0$, it suffices to show that $V(t+1) - V(t) = -p^T\Xi_t p \leq 0$. We use Lemmas 1 and 2 to obtain that
$$\begin{aligned} V(t+1) - V(t) &\leq -\lambda_2(\Xi_t)\|p - \pi_{H_0(\Xi_t\otimes I_m)}(p)\|^2 \\ &= -\lambda_2(\Xi_t)\|p\|^2 \leq 0. \end{aligned}$$
Clearly, $\|p\|$ is bounded by $\|p(0)\|$, it results from Lemma 6 that $\lambda_2(L_t) \geq c$ for some $c > 0$. Then, $\xi_i \geq hc(2 - h\cdot 2d_{\max}\alpha(0)) = c_1 > 0$ for $i \neq 1$ and $\xi_1 = 0$. Now, we have $V(t+1) - V(t) \leq -c_1\|p\|^2 < 0$ if $\|p\| \neq 0$. Hence, $\|p\| \to 0$ as $t \to \infty$. Note that $\sum_{i\in\mathcal{V}} x_i(t+1) = \sum_{i\in\mathcal{V}} x_i(t)$ in every step, which results in $\lim_{t\to\infty} x_i(t) = ((1)/(n))\sum_{i\in\mathcal{V}} x_i(0)$. Together with the radial unboundedness of V, consensus is achieved globally asymptotically. ■

For agents with dynamics (2), we propose the following protocol:
$$u_i(t) = -k_2 v_i(t) + k_3\sum_{j=1}^{n} g_{ij}\alpha_{ij}(x(t))(x_j(t) - x_i(t)) \quad (7)$$
where $k_2 > 0$ and $k_3 > 0$ are the coupling strengths, $i \in \mathcal{V}$.

Theorem 2: Consider a system consisting of n agents with dynamics (2). Under Assumption 1, suppose the communication graph is connected, and the following conditions for k_1, k_2, and k_3 are satisfied:

$$k_2 < \min\{2, k_1 + 1\} \tag{8}$$

$$k_3 < \min\left\{\frac{k_2(2-k_2)}{2d_{\max}\alpha(0)k_1(k_1-k_2+1)}, \frac{k_2}{d_{\max}\alpha(0)(k_1+1)}\right\}. \tag{9}$$

Then, protocol (7) globally asymptotically solves the consensus problem.

Proof: The discrete-time multiagent system (2) with protocol (7) can be rewritten as

$$\begin{aligned} p(t+1) &= p(t) + k_1 q(t)\\ q(t+1) &= q(t) - k_2 q(t) - k_3 L_t p(t). \end{aligned}$$

Consider the following function as a Lyapunov function candidate:

$$V(t) = \|k_2 p + k_1 q\|^2 + k_1\|q\|^2 + k_3(k_1 + 1 - k_2)W(t). \tag{10}$$

By Proposition 1, we have

$$\begin{aligned} &V(t+1) - V(t)\\ &\quad\le \|k_2 p(t+1) + k_1 q(t+1)\|^2 - \|k_2 p(t) + k_1 q(t)\|^2\\ &\qquad + k_1\|q(t+1)\|^2 - k_1\|q(t)\|^2\\ &\qquad + k_1^2 k_3(k_1 + 1 - k_2)q^T L_t q\\ &\qquad + 2k_1 k_3(k_1 + 1 - k_2)p^T L_t q\\ &\quad = q^T\big[k_1 k_2(k_2 - 2)I + k_1^2 k_3(k_1 + 1 - k_2)L_t\big]q\\ &\qquad + p^T\big[-2k_1 k_2 k_3 L_t + \big(k_1^2 + k_1\big)k_3^2 L_t^2\big]p\\ &\quad = -q^T\Xi_{1t}q - p^T\Xi_{2t}p. \end{aligned}$$

It is easy to see that $-q^T\Xi_{1t}q - p^T\Xi_{2t}p \le 0$ as long as the following inequalities hold for any $i \in \mathcal{V}$:

$$\begin{aligned} k_1 k_2(k_2 - 2) + k_1^2 k_3(k_1 + 1 - k_2)\lambda_i(L_t) &< 0\\ -2k_1 k_2 k_3 + \big(k_1^2 + k_1\big)k_3^2\lambda_i(L_t) &< 0. \end{aligned} \tag{11}$$

By Gerschgorin's theorem, it holds that $\lambda_i(L_t) \le \max_{i\in\mathcal{V}}\{2\sum_{j\in\mathcal{N}_i}\alpha_{ij}(x)\} \le 2d_{\max}\alpha(0)$. This implies that (8) and (9) ensure the validity of (11). Consequently, $V(t+1) - V(t) \le 0$. Since $\|k_2 p\| \le \|k_2 p + k_1 q\| + \|k_1 q\| \le 2\sqrt{V(t)} \le 2\sqrt{V(0)}$, by Lemma 6, $\lambda_2(L_t) \ge c$ for some $c > 0$. Let ξ_i', $i = 1, \ldots, n$ be the eigenvalues of Ξ_{2t}. From (9), $\xi_i' = [2k_1k_2k_3 - (k_1^2 + k_1)k_3^2\lambda_i(L_t)]\lambda_i(L_t) \ge [2k_1k_2k_3 - (k_1^2 + k_1)k_3^2 2d_{\max}\alpha(0)]c = c_2 > 0$, $i = 2, \ldots, n$, and $\xi_1' = 0$. That is, the zero eigenvalue of Ξ_{2t} is simple and all the other eigenvalues have a positive lower bound c_2. By Lemma 2, it holds that $V(t+1) - V(t) \le -k_1k_2(2-k_2)\|q\|^2 - c_2\|p\|^2$, implying that $V(t+1) - V(t) = 0$ if and only if $\|p\| = \|q\| = 0$.

Using $k_2 < k_1 + 1$, we have $\sqrt{V} \ge \|k_2 p + k_1 q\| \ge \|k_2 p\| - \|k_1 q\|$, and $\sqrt{V} \ge \sqrt{k_1}\|q\|$. Then, $\sqrt{V} + 2\sqrt{k_1 V} \ge k_2\|p\| + k_1\|q\|$. To show the radial unboundedness of V, it suffices to show that if $\|p\|^2 + \|q\|^2 \to \infty$, it holds that

$$V \ge \frac{\min\{k_2, k_1\}(\|p\|^2 + \|q\|^2)}{(1 + 2\sqrt{k_1})^2} \to \infty.$$

By Lyapunov stability theory, $\|p\|, \|q\| \to 0$ as $t \to \infty$ under arbitrary initial states. That is, both the position-like states and the velocity-like states of the agents globally asymptotically achieve consensus. ■

Remark 2: $\alpha_{ij}(\cdot)$ in this section can be different for different pairs of agents and each one satisfies Assumption 1. In this case, let $\alpha_{\max}(0) = \max_{i,j\in\mathcal{V}}\alpha(0)$. For system (1) with protocol (6), once the condition of h is changed to $h < ((1)/(d_{\max}\alpha_{\max}(0)))$, Theorem 1 can be valid. For multiagent system (2) with protocol (7), it holds that $W(t) = ((1)/(2))\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}} g_{ij} w_{ij}(\|x_{ij}(t)\|^2)$. By following similar lines to the proof of Theorem 2, we can obtain the same result as Theorem 2 except for replacing $\alpha(0)$ with $\alpha_{\max}(0)$ in (9).

D. Consensus in Networks With Distance-Dependent Connectivity

In networks with distance-dependent connectivity, the connectivity of the communication graph is possibly broken due to the evolution of agents. For reaching consensus, we hope that the connectivity can be maintained. In the following, we will seek a minimal level of an energy function related to the agents' initial states for the maintenance of connectivity. It is shown that under a dense distribution of the agents' initial states, consensus can be asymptotically reached. Before going into the details, we introduce Lemma 7.

Lemma 7: Under Assumption 2, if the communication graph $\mathcal{G}(t)$ is connected for any $t \ge 0$, then $\lambda_2(L_t)$ has a positive lower bound.

Proof: See the Appendix. ■

Theorem 3: Consider a system consisting of n agents with dynamics (1). Under Assumption 2, suppose $h < (1/((n-1)\alpha(0)))$, and there exists an $r \in [0, R^2)$, such that

$$W(0) < (n-1)w(R^2). \tag{12}$$

Then, protocol (6) asymptotically solves the consensus problem.

Proof: For any $t \ge 0$, the following holds from Proposition 1:

$$\begin{aligned} W(t+1) - W(t) &\le u^T L_t u + 2x^T L_t u\\ &= h^2 x^T L_t^3 x - 2hx^T L_t^2 x. \end{aligned}$$

By Gerschgorin's theorem, $\lambda_i(L_t) \le 2(n-1)\alpha(0)$. Together with $h < (1/((n-1)\alpha(0)))$, we have $h\lambda_i(L_t) - 2 < 0$. Hence, $h^2\lambda_i^3(L_t) - 2h\lambda_i^2(L_t) \le 0$, which implies that $W(t) \le W(0) < (n-1)w(R^2)$. From Proposition 1, $\mathcal{G}(t)$ is connected for any $t \ge 0$. It follows from Lemma 7 that $\lambda_2(L_t) \ge c > 0$ for some $c > 0$. Then, $\lambda_i(\Xi_t) \ge hc(2 - 2h(n-1)\alpha(0)) = c_1' > 0$ and $\lambda_1(\Xi_t) = 0$.

Consider $V(t) = \|p(t)\|^2$ as a Lyapunov function candidate. Through a similar process to the proof of Theorem 1, one has $V(t+1) - V(t) = p^T(-2hL_t + h^2L_t^2)p \le -c_1'\|p\|^2 < 0$ if $\|p\| \ne 0$. By the Lyapunov stability theory, $\|p\| \to 0$ as $t \to \infty$. That is, consensus is reached asymptotically. ■

Theorem 4: Consider a system consisting of n agents with dynamics (2). Under Assumption 2, suppose that k_1, k_2, and k_3 satisfy (8) and

$$k_3 < \min\left\{\frac{k_2(2-k_2)}{2(n-1)\alpha(0)k_1(k_1-k_2+1)}, \frac{k_2}{(n-1)\alpha(0)(k_1+1)}\right\}. \tag{13}$$

And there exists an $r \in [0, R^2)$, such that

$$\|k_2 p(0) + k_1 q(0)\|^2 + k_1\|q(0)\|^2 + (k_1 + 1 - k_2)k_3 W(0) < (k_1 + 1 - k_2)k_3(n-1)w(R^2). \quad (14)$$

Then, protocol (7) asymptotically solves the consensus problem.

Proof: Suppose condition (14) holds. Let $g_{ij} = 1$ for any $i, j \in \mathcal{V}$, (10) is considered as the Lyapunov function candidate. By following the same lines as in the proof of Theorem 2, we have $\lambda_2(L_t) \leq 2(n-1)\alpha(0)$, and the validity of (8) and (13) leads to the fact that $V(t+1) - V(t) = -q^T \Xi_{1t} q - p^T \Xi_{2t} p \leq 0$. Therefore, for any $t \geq 0$

$$V(t) \leq V(0) < \frac{1}{2}(k_1 + 1 - k_2)k_3(n-1)w(R^2).$$

As a result, $W(t) = (1/2)\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}} W_{ij}(t) < (n-1)w(R^2)$ for all $t \geq 0$. Proposition 1 shows that $\mathcal{G}(t)$ is always connected. From Lemma 7, $\lambda_2(L_t) \geq c$ for some $c > 0$. It follows that $\lambda_i(\Xi_{2t}) = [2k_1k_2k_3 - (k_1^2 + k_1)k_3^2\lambda_i(L_t)]\lambda_i(L_t) \geq [2k_1k_2k_3 - (k_1^2 + k_1)k_3^2 2(n-1)\alpha(0)]c = c_2' > 0$, $i = 2, \ldots, n$, and $\lambda_1(\Xi_{2t}) = 0$. As a result, $V(t+1) - V(t) \leq -k_1k_2(2-k_2)\|q\|^2 - c_2\|p\|^2$. Since $k_2 < 2$, $V(t+1) - V(t)$ is negative definite. It follows from the Lyapunov stability theory that $\|p\|,\ \|q\| \to 0$ as $t \to \infty$. That is, consensus is achieved asymptotically. ■

Remark 3: Under Assumption 2, if $r = R^2$, then $w(R^2) = 0$, and (12) and (14) can never be satisfied. Therefore, $r < R^2$ is indispensable in Theorems 3 and 4. Moreover, if r changes, (12) or (14) may be not valid. Although a smaller r makes $w(R^2)$ greater, but it does not hold that a smaller r makes it easier for the conditions to be satisfied.

IV. Consensus of Continuous-Time Multiagent Systems

A. Continuous Distance-Dependent Communication Weight

Like the discrete-time case, we utilize a function $\alpha(\cdot)$ to interpret the relationship between the transmission weight and the distance. To ensure the Lipschitz continuity of the control input, the previous assumptions for $\alpha(\cdot)$ are modified as follows.

Assumption 3: $\alpha(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is Lipschitz continuous and nonincreasing, $\alpha(0) < \infty$.

Assumption 4: $\alpha(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is Lipschitz continuous and nonincreasing, $\alpha(0) < \infty$, $\alpha(s) > 0$ if $s < R^2$, $\alpha(s) = 0$ if $s \geq R^2$, where $R \in \mathbb{R}_{>0}$ is a constant.

B. Consensus in Networks With Fixed Connectivity

For agents with dynamics (3), the following protocol is studied:

$$u_i = \sum_{j\in\mathcal{V}} g_{ij}\alpha_{ij}(x)(x_j - x_i) \quad (15)$$

where $i \in \mathcal{V}$. Since the communications between any two agents are symmetric, the cut-balance condition proposed in [29] is satisfied. Therefore, under Assumption 3, if the initial communication topology is connected, consensus can be achieved due to the fact that the connectivity is invariant. Actually, by differentiating the energy function $V = ((1)(2))\|p\|^2$ with respect to t, it is easy to obtain that $\|p\|$ is bounded by $\|p(0)\|$. As with the discrete-time case, the positive lower bound of $\lambda_2(L_x)$ that can be concluded from Lemma 6 finally guarantees the consensus result.

For agents with dynamics (4), the static consensus protocol is first studied

$$u_i = -kv_i + \sum_{j\in\mathcal{V}} g_{ij}\alpha_{ij}(x)(x_j - x_i) \quad (16)$$

where $k > 0$ is the feedback gain, $i \in \mathcal{V}$.

Theorem 5: Consider a system consisting of n agents with dynamics (4). Suppose the communication topology is connected and Assumption 3 is satisfied. Then, protocol (16) globally asymptotically solves the consensus problem. In particular, if the sum of the initial velocity-like state of each agent is zero, the average consensus problem is solved.

Proof: Consider the following Lyapunov function candidate:

$$V(x,v) = \|kx + v\|^2 + \|v\|^2 + \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}}\int_0^{\|x_i - x_j\|^2} a_{ij}(s)ds \quad (17)$$

where $a_{ij}(s) = g_{ij}\alpha(s)$. To show the radial unboundedness of $V(x,v)$, we let $\|x\|^2 + \|v\|^2 \to \infty$. Note that $\sqrt{V(x,v)} \geq \|kx + v\|$ and $\sqrt{V(x,v)} \geq \|v\|$, implying that $3\sqrt{V(x,v)} \geq \|kx\| - \|v\| + 2\|v\| = \|kx\| + \|v\|$, therefore

$$\begin{aligned} V(x,v) &\geq \frac{1}{9}(\|kx\| + \|v\|)^2 \\ &\geq \frac{1}{9}\min\{k^2, 1\}(\|x\|^2 + \|v\|^2) \to \infty. \end{aligned}$$

The derivative of $V(x,v)$ along the trajectories of the agents is given by

$$\begin{aligned} \dot{V} &= 2(kx + v)^T(kv - kv - L_x x) + 2v^T(-kv - L_x x) \\ &\quad + 2\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}} g_{ij}\alpha_{ij}(x)(x_i - x_j)^T(v_i - v_j) \\ &= -2kx^T L_x x - 2kv^T v \leq 0. \end{aligned}$$

Then, we obtain a positively invariant set, i.e., $\Omega(x,v) = \{x, v \mid V(x(t), v(t)) \leq V(x(0), v(0))\}$. It remains to show the compactness of Ω. Indeed, since V is continuous, $V^{-1}[0, V(x(0), v(0))]$ is closed. Recall the radial unboundedness of V, Ω is compact. Due to the Lipschitz continuity of the right-hand side of (16) and the fact that the system is autonomous, LaSalle's invariance principle can be used. Therefore, $\dot{V} \to 0$ as $t \to \infty$. Due to the connectivity of graph $\mathcal{G}$, Lemma 1 shows that x converges into M, and $\|v\| \to 0$ as $t \to \infty$.

Moreover, let $U(x,v) = \sum_{i\in\mathcal{V}} v_i + k\sum_{i\in\mathcal{V}} x_i$, we have $\dot{U} = -k\sum_{i\in\mathcal{V}} v + \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}}(x_j - x_i) + k\sum_{i\in\mathcal{V}} v = 0$. That is, $U(x^*, v^*) = U(x(0), v(0))$, where x^* is the consensus position-like state of each agent. Therefore, it can be obtained that $x^* = ((\sum_{i\in\mathcal{V}} v_i(0) + k\sum_{i\in\mathcal{V}} x_i(0))/nk)$.

If $\sum_{i\in\mathcal{V}} v_i(0) = 0$, one has $x^* = (1/n)\sum_{i\in\mathcal{V}} x_i(0)$, i.e., the average consensus is achieved. ■

Now, we consider the dynamic consensus protocol

$$u_i = \sum_{j\in\mathcal{V}} g_{ij}\alpha_{ij}(x)(v_j - v_i) + \sum_{j\in\mathcal{V}} g_{ij}\alpha_{ij}(x)(x_j - x_i) \quad (18)$$

where $i \in \mathcal{V}$.

Protocol (16) makes the velocity-like state of each agent vanish to zero for arbitrary initial value, and thus always keeps the distance between any two agents constant at steady state even if consensus is not reached. Hence, the compactness of Ω can be unconditionally guaranteed. However, each agent applying protocol (18) may obtain a nonzero velocity in the end, and as a result, the distance between the agents can be unbounded. To achieve global convergence, a condition for $\alpha(\cdot)$ is required to be appended.

Theorem 6: Consider a system consisting of n agents with dynamics (4). Suppose $\int_0^\infty \alpha(s)ds = \infty$ and Assumption 3 is satisfied. If the communication topology is connected, then protocol (18) globally asymptotically solves the consensus problem.

Proof: It is clear that x and v in the multiagent system (4) with (18) can be replaced by p and q. We consider the following energy-like function:

$$V(p,q) = \|q\|^2 + \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}}\int_0^{\|p_i-p_j\|^2} a_{ij}(s)ds. \quad (19)$$

Differentiating $V(p,q)$ along the trajectories of agents, one has

$$\begin{aligned}\dot{V}(p,q) &= 2q^T(-L_x p - L_x q)\\ &\quad + \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}} g_{ij}\alpha_{ij}(p)(p_i-p_j)^T(q_i-q_j)\\ &= -2q^T L_x q \le 0.\end{aligned}$$

Then, the set $\Omega = \{p,q|V(p,q) \le V(p(0),q(0))\}$ is positively invariant. To show the compactness of Ω, it suffices to show the boundedness of $\|p\|$ and $\|q\|$. Suppose $\|p(t)\| \to \infty$ as $t \to t^*$, $t^* > 0$ (t^* can be infinite). From Lemma 3, there exist a pair of agents, i.e., Agents i and j, such that $\|x_i - x_j\| \to \infty$ as $t \to t^*$. Due to the connectivity of the communication graph, there exists a path including $(i,i_1),\ldots,(i_s,j)$. Note that $\|x_i - x_j\| \le \|x_i - x_{i_1}\| + \cdots + \|x_{i_s} - x_j\|$. Therefore, there exists a constant $k \in \{1,\ldots,s\}$, such that $\|p_{i_k} - p_{i_{k+1}}\| = \|x_{i_k} - x_{i_{k+1}}\| \to \infty$. This yields

$$\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}}\int_0^{\|p_i-p_j\|^2} g_{ij}\alpha(s)ds \ge \int_0^{\|x_{i_k}-x_{i_{k+1}}\|^2}\alpha(s)ds \to \infty$$

as $t \to t^*$, which conflicts with $V(p,q) \le V(p(0),q(0))$ for all $t \ge 0$. Thus, $\|p(t)\|$ is bounded. It is clear that $\|q\|^2$ is bounded by $V(p(0),q(0))$, together with the closedness of $V^{-1}(0,V(p(0),q(0)))$, and it follows the radial unboundedness of $V(p,q)$ and the compactness of Ω. Due to the Lipschitz continuity of the right-hand side of (18) and the fact that the system is autonomous, we can employ LaSalle's invariance principle. Hence, p and q converge into $\{\dot{V}(p,q) = 0\}$ as $t \to \infty$. From the connectivity of the communication graph and Lemma 1, we have $\|q_i - q_j\| \to 0$ for any $i,j \in \mathcal{V}$, as $t \to \infty$. From Lemma 3, we have $\lim_{t\to\infty}\|\dot{p}\| = \lim_{t\to\infty} q = 0$, implying that both p and $\dot{q}$ are fixed at the steady state. It suffices to show that $\lim_{t\to\infty}\|\dot{q}\| = 0$. Hence, $\lim_{t\to\infty}\|-L_x p\| = \lim_{t\to\infty}\|\dot{q} + L_x q\| = 0$. Therefore, we obtain $\|p\| \to 0$ as $t \to \infty$. That is, consensus is asymptotically reached. ■

The condition for $\alpha(\cdot)$ is actually for the decay rate of the communication. It is clear that the faster $\alpha(\cdot)$ decays, the more difficult the condition is satisfied. In fact, when $\int_0^\infty \alpha(s)ds < \infty$, protocol (16) solves the consensus problem if the initial states of all agents satisfy an inequality. See Corollary 1.

Corollary 1: Consider a system consisting of n agents with dynamics (4). Under Assumption 1, suppose $\int_0^\infty \alpha(s)ds < \infty$, the communication graph $\mathcal{G}$ is connected and the following inequality holds:

$$\|q(0)\|^2 + \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}}\int_0^{\|p_i(0)-p_j(0)\|^2} g_{ij}\alpha(s)ds < k^*\int_0^\infty \alpha(s)ds \quad (20)$$

where k^* is the connectivity of graph $\mathcal{G}$. Then, protocol (18) solves the consensus problem asymptotically.

Proof: We still consider the energy-like function (19), and the next step is to show the compactness of $\Omega = \{p,q|V(p,q) \le V(p(0),q(0))\}$. Suppose $\|p\| \to \infty$, then there exist a pair of agents, i.e., Agents i and j, such that $\|p_i - p_j\| \to \infty$. By Lemma 4, there exist k^* disjoint paths between i and j. As the analysis in the proof of Theorem 6, in each path, there exist at least one pair of adjacent agents i_k and i_{k+1}, such that $\|p_{i_k} - p_{i_{k+1}}\| \to \infty$. Together with inequality (20), we have

$$\begin{aligned}V(p(0),q(0)) &= \|q(0)\|^2 + \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}}\int_0^{\|p_i(0)-p_j(0)\|^2} g_{ij}\alpha(s)ds\\ &< k^*\int_0^\infty \alpha(s)ds\\ &\le \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}}\int_0^{\|p_i-p_j\|^2} g_{ij}\alpha(s)ds\\ &\le V(p,q)\end{aligned}$$

which is a contradiction. Thus, $\|p\|$ is bounded for all $t \ge 0$. We now proceed as in the proof of Theorem 6. ■

Remark 4: All the above results can be extended to general cases. More specifically, $\alpha_{ij}(\cdot)$ can be different for different values of $(i,j) \in \mathcal{E}$. Each $\alpha_{ij}(s)$ is a continuous function of s and is unnecessary to be nonincreasing. In this case, the condition for $\alpha(\cdot)$ in Theorem 6 is replaced by the condition that there exists a spanning tree with $\mathcal{E}'$ as the corresponding set of edges, and $\int_0^\infty \alpha_{ij}(s)ds = \infty$ for any $(i,j) \in \mathcal{E}'$. If this is not valid, the initial states of the agents are required to satisfy the following inequality:

$$\begin{aligned}&\|v(0)\|^2 + \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}}\int_0^{\|x_i(0)-x_j(0)\|^2} a_{ij}\alpha(s)ds\\ &\quad < k^*\min_{(i,j)\in\mathcal{E}'}\int_0^\infty \alpha_{ij}(s)ds\end{aligned}$$

where $\mathcal{E}'$ is the set of edges associated with a spanning tree. The proof is similar to that of Theorem 6 and Corollary 1, we omit it here.

C. Consensus in Networks With Distance-Dependent Connectivity

Theorem 7: Consider a system consisting of n agents with dynamics (3). Under Assumption 4, suppose the following inequality holds:

$$\frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}}\int_0^{\|x_i(0)-x_j(0)\|^2}\alpha(s)ds < (n-1)\int_0^{R^2}\alpha(s)ds. \tag{21}$$

Then, protocol (15) solves the average consensus problem asymptotically.

Proof: Consider the Lyapunov function $V(x) = ((1)/(2))\|x\|^2$, we have $\dot{V}(x) = -x^T L_x x \leq 0$ and thus $\|x\| \leq \|x(0)\|$. It follows the compactness of the invariant set $\{x|\ V(x) \leq V(x(0))\}$. Due to the Lipschitz continuity of the right-hand side of (15) and the fact that the system is autonomous, LaSalle's invariance principle can be invoked. Thus, we have $\dot{V}(x) \to 0$ as $t \to \infty$. Then, x asymptotically converges into $H_0(L_x \otimes I_m)$ and $\lim_{t\to\infty}\mathcal{G}(t)$ is fixed.

We now consider the following function:

$$V_1(x) = \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}}\int_0^{\|x_i-x_j\|^2}\alpha(s)ds. \tag{22}$$

Differentiating $V_1(x)$ yields

$$\begin{aligned}\dot{V}_1(x) &= \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}}\alpha_{ij}(x)(x_i-x_j)^T(u_i-u_j)\\ &= 2x^T L_x u\\ &= -2\|\dot{x}\|^2 \leq 0.\end{aligned} \tag{23}$$

For any $t \geq 0$, it holds that

$$\begin{aligned}\frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}}\int_0^{\|x_i(t)-x_j(t)\|^2}\alpha(s)ds &\leq V_1(x(0))\\ &< (n-1)\int_0^{R^2}\alpha(s)ds.\end{aligned} \tag{24}$$

This implies that there are less than $n-1$ pairs of disconnected vertices in graph $\mathcal{G}(t)$. From Lemma 5, the communication graph is always connected. Therefore, $H_0(L_x\otimes I_m) = M$, i.e., consensus is achieved asymptotically. ∎

For agents with dynamics (4), we have the following result.

Theorem 8: Consider a system consisting of n agents with dynamics (4). Under Assumption 4, suppose the following inequality holds:

$$\begin{aligned}\|v(0)\|^2 + \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}}\int_0^{\|x_i(0)-x_j(0)\|^2}\alpha(s)ds\\ < (n-1)\int_0^{R^2}\alpha(s)ds.\end{aligned} \tag{25}$$

Then, protocol (16) solves the consensus problem asymptotically.

Proof: By following the same lines as in the proof of Theorem 5, it holds that x converges into $H_0(L_x \otimes I_m)$ and $\|v\| \to 0$ as $t \to \infty$, which implies that graph $\mathcal{G}$ is finally fixed. For obtaining a relaxed criterion for consensus, we consider the energy function (19) by replacing p and q with x and v. It is easy to know $\dot{V} = -2k\|v\|^2 \leq 0$. Therefore, for any $t \geq 0$

$$\begin{aligned}\frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}}\int_0^{\|x_i(t)-x_j(t)\|^2}\alpha(s)ds &\leq V(t)\\ &\leq V(0)\\ &< (n-1)\int_0^{R^2}\alpha(s)ds\end{aligned}$$

which implies that there are less than $n-1$ pairs of disconnected vertices in graph $\mathcal{G}(t)$. Together with Lemma 5, graph $\mathcal{G}(t)$ is always connected. Lemma 1 shows that $H_0(L_x \otimes I_m) = M$. Hence, all the agents achieve consensus asymptotically. ∎

V. Applications to Opinion Dynamics and Formation Control

A. Applications to Opinion Dynamics

In this section, we consider the consensus problem of opinion formation among a group of agents. More specifically, each agent keeps a real number as its opinion and updates it by taking a weighted average for the opinions of its neighbors. Any two agents interact with each other if and only if their opinion difference is less than a specified bound, which is called the confidence bound.

For discrete-time opinion dynamics, the following opinion evolution model is considered:

$$x_i(t+1) = \sum_{j\in\mathcal{V}} w_{ij}(x)x_j(t) \tag{26}$$

where $w_{ij} \geq 0$ denotes the weight between Agents i and j, and $\sum_{j\in\mathcal{V}} w_{ij} = 1$ for any $i \in \mathcal{V}$. It is clear that the multiagent system (1) with protocol (6) can be considered as an opinion formation problem if $\alpha(\cdot)$ satisfies Assumption 2. Since each agent will consider its own opinion in a positive way, to make this hold, we assume $(n-1)h < 1$. Then, model (26) can be rewritten as

$$x_i(t+1) = \left(1 - h\sum_{j\neq i}\alpha_{ij}\right)x_i(t) + h\sum_{j\neq i}\alpha_{ij}x_j(t). \tag{27}$$

Let

$$\alpha(s) = \begin{cases}1, & 0 \leq s < R^2\\ 0, & s \geq R^2\end{cases} \tag{28}$$

then Assumption 2 is satisfied. According to Theorem 3, the agents achieve the average consensus of opinions if (12) holds.

In fact, the distribution of the agents' initial states and the number of agents are important factors affecting the evolution of the network connectivity. In the following, we will present a more relaxed condition in the circumstance that the initial opinions are symmetrically distributed.

Consider a system consisting of n agents, Agent i keeps a real number x_i as its opinion. Without loss of generality, assume that $x_i \leq x_j$ if $i \leq j$. We say that the states are symmetrically distributed if there exists a real number x_0, such that $x_0 = ((x_i + x_j)/2)$ for any $i+j=n$. Then, Proposition 2 holds.

Proposition 2: Consider model (27) with $n \geq 4$ agents, suppose that the initial states of the agents are symmetrically distributed. For any $t \geq 0$, if there are less than $2n-3$ pairs of disconnected agents, then the communication graph $\mathcal{G}(t)$ is connected.

Proof: See the Appendix. ■

Theorem 9: Consider model (27) with n symmetrically distributed opinions at the initial time and $h < (1/(\alpha(0)(n-1)))$. Then, the following statements hold.

1) For $2 \leq n \leq 3$, the average consensus of the opinions is achieved if and only if the initial communication graph is connected.
2) For $n \geq 4$, the average consensus of the opinions is achieved if there exists an $r \in [0, R^2)$, such that

$$W(0) < (2n-3)w(R^2). \tag{29}$$

Proof:

1) From the analysis in the proof of Theorem 7, we know that once the connectivity of the communication topology is preserved, the system can always reach consensus. In the following proof, for simplicity, we let $\triangle e = e(t+1) - e(t)$ for any $e \in \mathcal{R}$. For $n=2$, let x_1 and x_2 be the two agents' opinions and $e = x_2 - x_1$, then $\triangle e = \triangle x_2 - \triangle x_1 = -2h\alpha_{12}e(t)$. Suppose $|e(0)| < R$. Note that $\triangle e > 0$ if $e(t) < 0$ and $\triangle e < 0$ if $e(t) > 0$, which implies that $|e(t)|$ is decreasing of t. Therefore, the two agents always keep connected. If $e(0) \geq R$, then $\triangle e = 0$, and consensus will never be reached. For $n=3$, let x_1, x_2, and x_3 be the three opinions and $x_1 \leq x_2 \leq x_3$. From Lemma 10, we have $x_2 = ((x_1+x_3)/2)$ and $\triangle x_2 = 0$ for any $t \geq 0$. Then, $\triangle x_1 = \alpha_{12}(x_2(t) - x_1(t)) + \alpha_{13}(x_3(t) - x_1(t)) = (\alpha_{12} + 2\alpha_{13})(x_2(t) - x_1(t))$. Similarly, we have $\triangle x_3 = (\alpha_{23} + 2\alpha_{13})(x_2(t) - x_3(t))$. Let $e_1 = x_1 - x_2$ and $e_2 = x_3 - x_2$, it follows that $\triangle e_1 = -(\alpha_{12} + 2\alpha_{13})e_1(t)$ and $\triangle e_2 = -(\alpha_{23} + 2\alpha_{13})e_2(t)$. If $|e_1(0)| < R$ and $|e_2(0)| < R$, then $|e_1(t)|$ and $|e_2(t)|$ are decreasing for any $t \geq 0$. That is, the connectivity of the communication graph is maintained. Suppose the initial communication graph is not connected. Without loss of generality, assume $|e_1(0)| > R$, then $\triangle x_1 = 0$, together with $\triangle x_2 = 0$; one has $\triangle e_1 = 0$, and consensus cannot be reached.
2) By employing Proposition 2, the proof is similar to the one of Theorem 3. ■

In the continuous-time case, agents (3) with protocol (15) can be considered as a smoothed model for opinion dynamics if the following $\alpha(\cdot)$ is applied:

$$\alpha(s) = \begin{cases} c, & 0 \leq s < (R-\varepsilon)^2 \\ f(s), & (R-\varepsilon)^2 \leq s < R^2 \\ 0, & s \geq R^2 \end{cases} \tag{30}$$

where $c > 0$ is the communication weight between neighbors, $R > 0$ is the bound of confidence, $f(s)$ is a nonincreasing and Lipschitz continuous function of s in $[(R-\varepsilon)^2, R^2]$, and $f((R-\varepsilon)^2) = c$, $f(R^2) = 0$. This smoothed model makes such an assumption that when the opinion difference between Agents i and j exceeds the confidence bound, the information transmission between them vanishes smoothly. In [17], ε is set by a sequence that $f(s)$ closely depends on, i.e., $f(s) = (c/\varepsilon)(R - \sqrt{s})$, this model, which is called an ε approximation for the Hegselmann and Krause model. It is obvious that Theorem 7 can be applied to this model. Therefore, the average consensus can be reached if the initial states of agents satisfy (21).

As with Theorem 9, a more relaxed condition for continuous-time opinion formation model can be obtained if the initial distribution of the agents' states is symmetric.

Theorem 10: Consider the multiagent system (3) with protocol (15), the communication weight $\alpha(\cdot)$ is set by (30). Suppose there are n symmetrically distributed opinions at the initial time. Then, the following statements hold.

1) For $2 \leq n \leq 3$, the average consensus of the opinions is achieved if and only if the initial communication graph is connected.
2) For $n \geq 4$, the average consensus of the opinions is achieved if the following inequality holds:

$$\frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}}\int_0^{\|x_i(0)-x_j(0)\|^2}\alpha(s)ds < (2n-3)\int_0^{R^2}\alpha(s)ds. \tag{31}$$

B. Applications to Formation Control

Now, we apply the main results to stabilizing multiple continuous-time agents to form a specific geometric shape in the plane. Suppose that the control input of each agent is applied as its acceleration [22]. That is, all the agents are governed by double-integrator dynamics (4).

In the literature, formation control problems are often solved by either distance-based approaches [2], [3] or displacement-based approaches [4], [5]. In both cases, the control input of each agent can be designed as one of the following two forms:

$$u_i = -\nabla_{x_i}\psi - kv_i, \quad i \in \mathcal{V} \tag{32}$$

$$u_i = -\nabla_{x_i}\psi + \sum_{j\in\mathcal{V}} a_{ij}(v_j - v_i), \quad i \in \mathcal{V} \tag{33}$$

where $\psi = \psi(\|x_{ij}\|^2)$ is a nonnegative and continuously differentiable potential function. It is easy to see that if we let $\psi = (1/4)\sum_{i,j\in\mathcal{V}}\int_0^{\|x_i-x_j\|^2} a_{ij}(s)ds$, (32) and (33) are equivalent to (16) and (18), respectively. Therefore, the analysis approaches of consensus can also be used to obtain conditions for the stability of formation. The main difference between consensus and formation is whether the desired distance between the adjacent agents is zero or a nonzero constant. Moreover, one can notice that if $\psi = 0$ and $a_{ij}(\cdot)$ satisfies Assumption 3, the multiagent system (4) with protocol (33) becomes the C-S flocking model [34]. In the following, we will show that our results corresponding to fixed network connectivity can be applied to achieving distance-based formation, while the results corresponding to distance-dependent network connectivity can be utilized to accomplish displacement-based formation.

1) Distance-Based Formation: Suppose the communication weight a_{ij} satisfies Assumption 3. Given a realization $x^* = (x_1^{*T}, \dots, x_n^{*T})^T$, the desired formation E^* is denoted by $E^* = \{\|x_i - x_j\| = \|x_i^* - x_j^*\| = d_{ij}, v_i = v_j, \forall i, j \in \mathcal{V}\}$. Let $\psi = (1/4)\sum_{i,j\in\mathcal{V}} \int_0^{(\|x_i - x_j\|^2 - d_{ij}^2)^2} a_{ij}(\sqrt{s} + d_{ij}^2)ds$. Then, protocol (33) can be rewritten as

$$u_i = \sum_{j\in\mathcal{V}} a_{ij}\left(\|x_i - x_j\|^2 - d_{ij}^2\right)(x_j - x_i) + \sum_{j\in\mathcal{V}} a_{ij}(v_j - v_i) \tag{34}$$

where $a_{ij} = g_{ij}\alpha(|(\|x_{ij}\|^2 - d_{ij}^2)| + d_{ij}^2) \le g_{ij}\alpha(\|x_{ij}\|^2)$. As with the analysis approach of Theorem 6, by constructing the energy-like function $V = 2\psi + \|v\|^2$, we can obtain that if $\int_0^{\infty} \alpha(\sqrt{s} + d_{ij}^2)ds = \infty$ for any $i, j \in \mathcal{V}$, then $\|p_i - p_j\| = \|x_i - x_j\|$ is bounded for any $i, j \in \mathcal{V}$. It follows the compactness of $\{(p^T, q^T)^T | V \le V(0)\}$. This guarantees that all agents' states asymptotically converge into $E = \{\dot{V} = 0\} = \{\nabla_{x_i}\psi = 0, v_i = v_j, \forall i, j \in \mathcal{V}\}$. Note that $E^* \subset E$ and $E^* \ne E$. From [3], the global stability of formation E^* cannot be guaranteed. However, the local stability may be obtained if the graph $(\mathcal{G}, x^*)$ is rigid. The detailed definition of graph rigidity can be found in [3]. We leave to future work the strict proof of local stability of the distance-based formation.

2) Displacement-Based Formation: Now, we suppose that a_{ij} satisfies Assumption 4, which implies that every agent has a sensing range $R > 0$. Let h_{ij} be the desired displacement between Agents i and j. Assume that $\|h_{ij}\| < R$ for any $i, j \in \mathcal{V}$. Note that if this assumption does not hold for some pair of agents, these two agents will lose the ability to communicate with each other when the desired displacement between them is reached, which results in the failure of formation maintaining. Let $\psi = (1/4)\sum_{i,j\in\mathcal{V}} \int_0^{\|x_i - x_j - h_{ij}\|^2} \alpha((\sqrt{s} + h^*)^2)ds$, where $h^* = \max_{i,j\in\mathcal{V}} \|h_{ij}\|$. Then, protocol (32) can be written as

$$u_i = -kv_i + \sum_{j\in\mathcal{V}} \alpha(z)(x_j - x_i + h_{ij}) \tag{35}$$

where $z = (\|x_i - x_j - h_{ij}\| + h^*)^2$. Due to the fact that $\alpha(\cdot)$ is nonincreasing, we have $\alpha(z) \le \alpha(\|x_i - x_j\|^2)$. Hence, whenever $\alpha(z) > 0$, it holds that $\|x_i - x_j\| \le \sqrt{z} < R$, which implies that protocol (35) is achievable. Let $y_i = x_i - h_i$ for $i \in \mathcal{V}$, where $\{h_1, \dots, h_n\}$ is a specific set, such that $h_i - h_j = h_{ij}$ for any $i, j \in \mathcal{V}$. It is easy to see that $y_i - y_j = 0$ if and only if $x_i - x_j = h_{ij}$. Therefore, the formation control problem is transformed into the consensus problem of the following system:

$$\begin{aligned} \dot{y}_i &= v_i \\ \dot{v}_i &= -kv_i + \sum_{j\in\mathcal{V}} \alpha((\|y_{ij}\| + h^*)^2)(y_j - y_i), \quad i \in \mathcal{V}. \end{aligned}$$

From Theorem 8, we obtain the following condition for formation:

$$\begin{aligned} &\|v(0)\|^2 + \frac{1}{2}\sum_{i,j\in\mathcal{V}} \int_0^{\|y_i(0) - y_j(0)\|^2} \alpha((\sqrt{s} + h^*)^2)ds \\ &\quad < (n-1)\int_0^{(R-h^*)^2} \alpha((\sqrt{s} + h^*)^2)ds \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\|v(0)\|^2 + \frac{1}{2}\sum_{i,j\in\mathcal{V}} \int_{h^{*2}}^{(\|x_i(0) - x_j(0) - h_{ij}\| + h^*)^2} \alpha(s)ds \\ &\quad < (n-1)\int_{h^{*2}}^{R^2} \alpha(s)ds. \end{aligned} \tag{36}$$

It should be mentioned that if h^* is close to R, it is difficult to satisfy condition (36). In fact, it is easy to imagine that the network connectivity is difficult to be maintained if the desired distance between the adjacent agents is close to their sensing range. Hence, the formation is more likely to be achieved if h^* is much less than R.

VI. Simulation Examples

This section presents several simulation examples to validate the effectiveness of the obtained theoretical results. We first design three examples to illustrate the case when the connectivity of the communication graph is fixed. Assume that the communication weight between Agents i and j is the same as the one of the C-S flocking model [34], which can be written as

$$a_{ij} = \frac{H}{(1 + \|x_i - x_j\|^2)^{\beta}} \tag{37}$$

where $H > 0$ and $\beta \ge 0$ are system parameters. That is, $\alpha(s) = (H/((1+s)^{\beta}))$ and $\mathcal{G}$ is a complete graph. Then, both Assumptions 1 and 3 are satisfied.

Example 1: Consider a group of agents with dynamics (1) and protocol (6), let $n = 30$, $H = 1$, and $\beta = 3$. Due to Theorem 1, if $h < (1/(n-1))$, consensus can be asymptotically reached under arbitrary initial states. Figs. 3 and 4 show the trajectories of the agents with $h = (2/n)$ and $h = (1/n)$, respectively, under the same

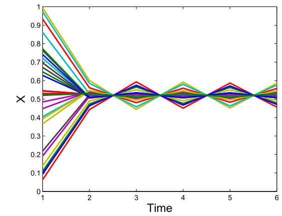


Fig. 3. Trajectories of agents with protocol (6) and $h = (2)/(n)$.

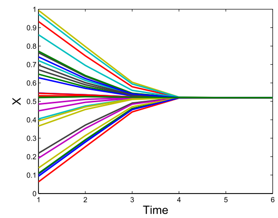


Fig. 4. Trajectories of agents with protocol (6) and $h = (1)/(n)$.

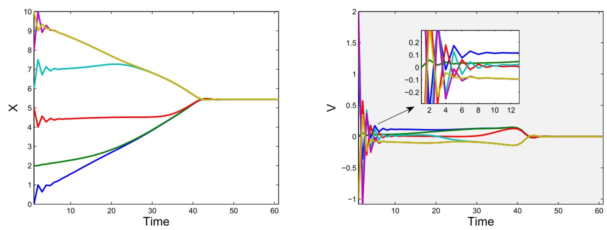


Fig. 5. Trajectories of agents with protocol (7) when $k_3 = 0.14$.

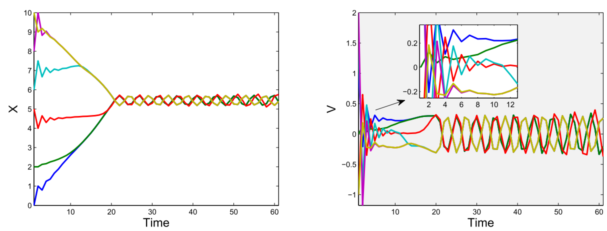


Fig. 6. Trajectories of agents with protocol (7) when $k_3 = 0.3$.

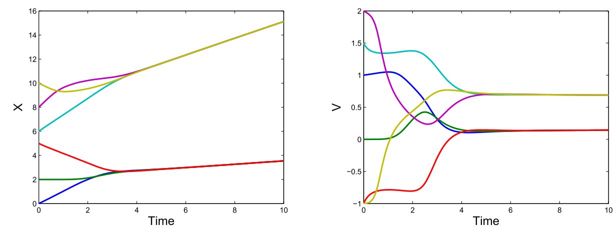


Fig. 7. Trajectories of agents with dynamics (4) and protocol (18), $\beta = 2$.

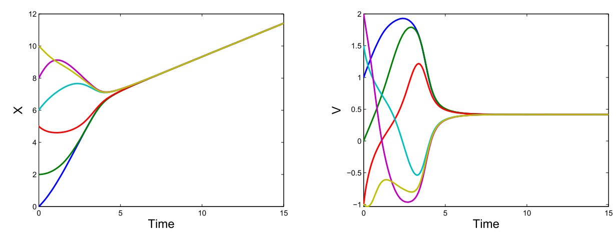


Fig. 8. Trajectories of agents with dynamics (4) and protocol (18), $\beta = 1$.

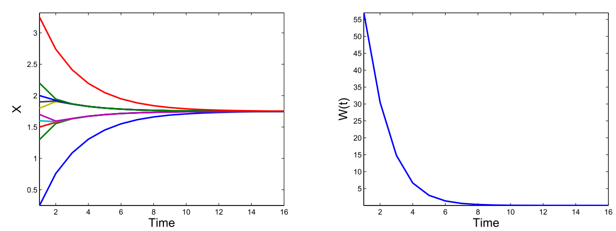


Fig. 9. Discrete-time opinion formation.

initial states. Fig. 4 shows that consensus is reached, therefore consisting with the result in Theorem 1.

Example 2: Now, we consider a group of agents with dynamics (2) and protocol (7). Let $n = 6$, $H = 1$, $\beta = 1$, $k_1 = 1$, and $k_2 = 1.5$. Then, condition (8) is satisfied. The initial states of the agents are set by $x(0) = (0, 2, 5, 6, 8, 10)^T$, $v(0) = (1, 0, -1, 1.5, 2, -1)^T$. When we set $k_3 = 0.14$, condition (9) is satisfied. By Theorem 2, the agents achieve consensus asymptotically under arbitrary initial states. Fig. 5 shows that consensus is reached. When we set $k_3 = 0.3$, condition (9) is invalid. Fig. 6 shows that consensus is not reached.

Example 3: Consider a system consisting of six agents with dynamics (4) and protocol (18). The initial position-like states and velocity-like states are set the same as the one in Example 2. Note that

$$\int_0^\infty \frac{H}{(1+s)^\beta} ds = \begin{cases} \frac{H}{1-\beta}(1+s)^{1-\beta}\big|_0^\infty, & \beta \neq 1 \\ H\ln(1+s)\big|_0^\infty, & \beta = 1. \end{cases}$$

Therefore, $\int_0^\infty \alpha(s)ds = \infty$ if $\beta \leq 1$. According to Theorem 6, consensus is asymptotically reached. Otherwise, $\int_0^\infty \alpha(s)ds < \infty$ if $\beta > 1$. Let $H = 1$ and $\beta = 2$, Fig. 7 shows that consensus of agents' states is not reached. When we set $H = 1$ and $\beta = 1$, consensus is asymptotically achieved, as shown in Fig. 8.

Now, we present two examples to show the effectiveness of the results when the network connectivity is distance-dependent. That is, Assumption 2 or 4 is satisfied.

Example 4: Consider model (27) with 20 agents. Their initial opinions are symmetrically distributed, i.e., $x(0) = (0.25, 1.3, 1.5, 1.6, 1.7, 1.8, 1.9, 2, 2.2, 3.25)^T$. Suppose $R = 1.5$ and $\alpha(s) = 1.5$ if $s < R^2$ and $\alpha(s) = 0$ if $s \geq R^2$, $h = (1/(\alpha(0)n))$. Let $r = 0.02$, it is computed that $W(0) = 56.97 < (2n-3)w(R^2) = 57.12$. From Theorem 9, consensus can be asymptotically reached, as shown in Fig. 9.

Example 5: Consider an example of the smoothed opinion formation model, in which $\alpha(\cdot)$ is set in form (30). Then, Assumption 4 is satisfied. Assume the opinions are set symmetrically distributed at the initial time, i.e., $x(0) = (1, 1.7, 1.85, 1.9, 1.95, 2.05, 2.1, 2.15, 2.3, 3)^T$. Let $R = 1$, $\varepsilon = 0.01$, and $c = 1$, which implies that $f(s) = 100(1-\sqrt{s})$. Note that $V_1 = 16.76 < (2n-3)\int_0^{R^2} \alpha(s)ds = 16.83$, where V_1 is in form (22). By Theorem 10, the average consensus problem is solved. Fig. 10 shows the trajectories of the opinions and V_1. In this example, the condition presented in [18] is obviously invalid, since there exist two agents sharing no common neighbors.

The following two examples are for illustrating the effectiveness of our protocols for the formation control in networks with fixed connectivity and distance-dependent connectivity, respectively. We will consider six agents with dynamics (4) moving in the plane. The desired formation is a regular hexagon with the side of length 1.

Example 6: Consider the case when the network connectivity is fixed, we assume that the communication weight a_{ij} is in

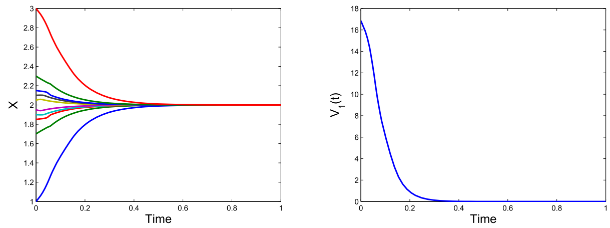


Fig. 10. Continuous-time opinion formation.

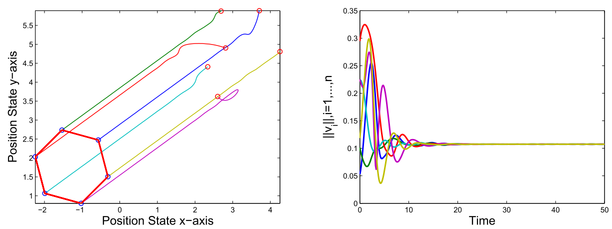


Fig. 11. Six agents with protocol (34) converging to a regular hexagon.

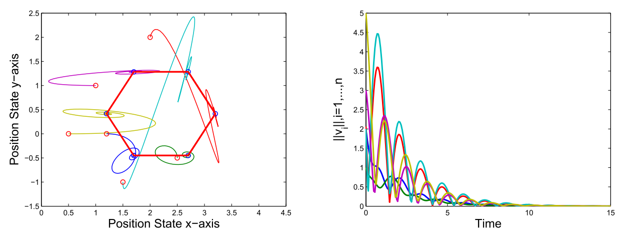


Fig. 12. Six agents with protocol (35) converging to a regular hexagon.

form (37) and $(\mathcal{G}, x^*)$ is complete. It is easy to see that if $H > 0$, $\beta \leq 2$, it holds that

$$\int_0^\infty \alpha\big(\sqrt{s} + d_{ij}^2\big)ds = \int_0^\infty \frac{H}{\big(\sqrt{s} + d_{ij}^2 + 1\big)^\beta} ds = \infty.$$

Since a complete graph is globally rigid, the formation can be locally asymptotically achieved. By setting $H = 1$ and $\beta = 2$, Fig. 11 shows the trajectories of six double-integrator agents implementing protocol (34). In Fig. 11, the red points represent the initial position states of the agents and the blue points represent the current position states of them. It can be seen that at the steady state, all the agents achieve a common velocity state and their position states form a regular hexagon with the side of length 1.

Example 7: Consider the case when the network connectivity is distance-dependent, protocol (35) is used. We assume that $\alpha(\cdot)$ is in form (30), where $f(s) = \frac{c}{\varepsilon}(R - \sqrt{s})$, $c = 1$, and $\varepsilon = 0.1$. Let the sensing range of each agent be $R = 6$. By choosing proper initial states satisfying condition (36), Fig. 12 shows that protocol (35) successfully solves the formation problem.

VII. Conclusion

This paper solved the consensus problem of multiagent systems in two types of distance-dependent communication networks. In the first type of networks, the connectivity is fixed while the transmission volume is distance-dependent. It was found that by applying the proposed protocols to discrete-time multiagent systems, agents achieve consensus if the control gains satisfy some specified conditions. For the continuous-time case, the protocols can be globally effective if the decay rate of the communication weight is less than a given bound. In the second type of networks, the motion of agents influences the network connectivity and the transmission volume, simultaneously. We obtained that for both discrete-time and continuous-time multiagent systems, the connectivity of the communication network can be maintained if the agents stay close enough to each other at the initial time. As a result, consensus can be achieved asymptotically. The main results were subsequently applied to solving the consensus problem of opinion dynamics and the formation control problem. Finally, the numerical simulations were performed to validate the effectiveness of the obtained theoretical results.

The future work will focus on some other interesting topics with distance-dependent communications in distributed coordination of multiagent systems, such as the containment problem, the tracking problem, the case with directed communication topology, and the time delay.

Appendix
Proofs of Lemmas and Propositions

The proof of Lemma 4 is based on Lemmas 8 and 9.

Lemma 8 (Menger's Theorem [39]): If x and y are vertices of a graph $\mathcal{G}$ and $(x, y) \notin \mathcal{E}(\mathcal{G})$, then the minimum size of an $x, y-$cut equals the maximum number of pairwise internally disjoint $x, y-$paths.

Lemma 9 [39]: Deletion of an edge reduces connectivity by at most 1.

Proof of Lemma 4: Assume that there exist a pair of agents i and j, and the maximum number of disjoint paths between them is $l < k^*$. We first consider the case when $(i, j) \notin \mathcal{E}(\mathcal{G})$, from Lemma 8, the minimum size of an i, $j-$cut in graph $\mathcal{G}$ is l. This means that the minimum size of a vertex set disconnecting i and j is l. Therefore, $\kappa(\mathcal{G}) \leq l < k^*$, which is a contradiction. For the case when $(i, j) \in \mathcal{E}(\mathcal{G})$. Let $\mathcal{G}' = \mathcal{G} - \{(i, j)\}$, from Lemma 9, $\kappa(\mathcal{G}') \geq \kappa(\mathcal{G}) - 1$. By Menger's theorem, the minimum size of an i, $j-$cut in graph $\mathcal{G}'$ is $l - 1$. Hence, $\kappa(\mathcal{G}') \leq l - 1$. Then, $\kappa(\mathcal{G}) \leq \kappa(\mathcal{G}') + 1 \leq l < k^*$, which conflicts with $\kappa(\mathcal{G}) = k^*$. ■

Proof of Lemma 5: We prove the contrapositive of the statement. Without loss of generality, suppose that $\mathcal{G}$ has r connected components, $\mathcal{V}_1, \ldots, \mathcal{V}_r$ are the corresponding sets of vertices, and $|\mathcal{V}_1| \leq \cdots \leq |\mathcal{V}_r|$. Let $\mathcal{V}_p$ be the first set which has more than one element. That is, $p = \min_{|\mathcal{V}_i| \geq 2} i$. Let $f(r)$ denote the minimal number of pairs of disconnected vertices, $n_i = |\mathcal{V}_i|$. We have $f(r) = C_n^2 - C_{n_p}^2 - C_{n_{p+1}}^2 - \cdots - C_{n_r}^2$. Combining $\mathcal{V}_p$ and $\mathcal{V}_{p+1}$, it follows that $f(r-1) \leq C_n^2 - C_{n_p+n_{p+1}}^2 - C_{n_{p+2}}^2 - \cdots - C_{n_r}^2$. Thus

$$f(r) - f(r-1) \geq C_{n_p+n_{p+1}}^2 - C_{n_p}^2 - C_{n_{p+1}}^2 > 0.$$

Consequently, $f(r)$ is an increasing function of r. Since the graph is not connected, one has $r > 1$. Thus, $f(r) \geq f(2)$. Recall that $f(2) = \min\{n_1 n_2\} = \min\{n_1(n - n_1)\} = n - 1$. Therefore, $f(r) \geq n - 1$. ■

Proof of Lemma 6: If $\|p(t)\|$ is upper bounded, we obtain the upper bound B of $\|x_i - x_j\|$ for any $i, j \in \mathcal{V}$ from Lemma 3. Let e denote the eigenvector associated with $\lambda_2(L_t)$, due to the fact that $\alpha(s)$ is nonincreasing of s, we have

$$\begin{aligned}\lambda_2(L_t) = \frac{e^T L_t e}{e^T e} &= \frac{\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}} g_{ij}\alpha_{ij}\|e_i - e_j\|^2}{2e^T e}\\ &\geq \alpha(B)\cdot\frac{\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}} g_{ij}\|e_i - e_j\|^2}{2e^T e}\\ &= \alpha(B)\cdot\frac{e^T\bar{L}e}{e^T e} \geq \alpha(B)\lambda_2(\bar{L})\end{aligned}$$

where $\bar{L}$ is the Laplacian matrix of graph $\bar{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, G)$. Since graph $\mathcal{G}$ is connected, $\lambda_2(\bar{L}) > 0$. Thus, $\lambda_2(L_t)$ has a positive lower bound. ■

Proof of Lemma 7: By Assumption 2, $a_{ij} \geq \alpha^* > 0$ if $(i, j) \in \mathcal{E}$. For a fixed time $t \geq 0$, through a similar process to the proof of Lemma 6, we have $\lambda_2(L_t) \geq \alpha^*\lambda_2(\bar{L}_t)$, where $\bar{L}_t$ is the Laplacian matrix of graph $\bar{\mathcal{G}}(t) = (\mathcal{V}, \mathcal{E}, G(t))$. Note that $G(t)$ is an adjacency matrix with elements 0 and 1. Since graph $\mathcal{G}(t)$ is connected for any $t \geq 0$, the number of possible $G(t)$ is finite. Moreover, for each possible $G(t)$, $\lambda_2(\bar{L}_t) > 0$. It follows that $\lambda_2(\bar{L}_t) \geq \lambda^*$ for some $\lambda^* > 0$. Therefore, $\lambda_2(L_t) \geq \alpha^*\lambda^* > 0$.

Proof of Proposition 1:

1) The sufficiency is obvious, we only prove the necessity. Suppose $W = 0$, then for any $(i, j) \in \mathcal{E}$, $w(\|x_{ij}(t)\|^2) = W_{ij} = 0$, implying that $\|x_i - x_j\| = 0$. For any $i, j \in \mathcal{V}$, since graph $\mathcal{G}$ has a spanning tree, it follows that $\|x_i - x_j\| \leq \sum_{(i_k, i_{k+1})\in P}\|x_{i_k} - x_{i_{k+1}}\| = 0$, where $P = \{(i, i_1), (i_1, i_2), \ldots, (i_s, j)\}$ is the path from i to j. Hence, $x_i = x_j$.
2) Suppose that $0 < z_1 < z_2$. We study this problem in the following three cases.

   Case 1: $z_1 < z_2 < r$. Then, $w(z_2) - w(z_1) = \alpha(r)(z_2 - z_1) \geq 0$.

   Case 2: $z_1 < r \leq z_2$. It follows that $w(z_2) - w(z_1) \geq \alpha(r)r - \alpha(r)z_1 \geq 0$.

   Case 3: $r < z_1 < z_2$. If $\lfloor z_2/r\rfloor > \lfloor z_1/r\rfloor$, then

$$\begin{aligned}&w(z_2) - w(z_1)\\ &\quad\geq \alpha\left(\left\lfloor\frac{z_1}{r}\right\rfloor r + r\right)r + \alpha\left(\left\lceil\frac{z_2}{r}\right\rceil r\right)\left(z_2 - \left\lfloor\frac{z_2}{r}\right\rfloor r\right)\\ &\qquad - \alpha\left(\left\lceil\frac{z_1}{r}\right\rceil r\right)\left(z_1 - \left\lfloor\frac{z_1}{r}\right\rfloor r\right)\\ &\quad\geq \alpha\left(\left\lceil\frac{z_2}{r}\right\rceil r\right)\left(z_2 - \left\lfloor\frac{z_2}{r}\right\rfloor r\right) \geq 0.\end{aligned}$$

   If $\lfloor z_2/r\rfloor = \lfloor z_1/r\rfloor$, we have $z_2 - \lfloor z_2/r\rfloor r > z_1 - \lfloor z_1/r\rfloor r$. Hence, $w(z_2) - w(z_1) = \alpha(\lceil z_2/r\rceil r)(z_2 - \lfloor z_2/r\rfloor r) - \alpha(\lceil z_1/r\rceil r)(z_1 - \lfloor z_1/r\rfloor r) \geq 0$.
3) For any $t \geq 0$, note that $\alpha_{ij}(x(t+1)) \leq \alpha_{ij}(x(t))$ if $\|x_{ij}(t+1)\| \geq \|x_{ij}(t)\|$ and $\alpha_{ij}(x(t+1)) \geq \alpha_{ij}(x(t))$ if $\|x_{ij}(t+1)\| \leq \|x_{ij}(t)\|$. Therefore, we always have

$$\begin{aligned}&W_{ij}(t+1) - W_{ij}(t)\\ &\quad\leq \alpha_{ij}(x(t))(\|x_{ij}(t+1)\|^2 - \|x_{ij}(t)\|^2).\end{aligned}$$

   Then, (5) can be obtained.
4) For any $r \leq z < \infty$, $\alpha(\cdot)$ is Riemann integral on $[0, z]$, since it is monotonous and bounded by $\alpha(0)$. Then, we have

$$\begin{aligned}&\int_0^z \alpha(s)ds\\ &\quad= \int_0^r \alpha(s)ds + \cdots + \int_{(\lfloor\frac{z}{r}\rfloor - 1)r}^{\lfloor\frac{z}{r}\rfloor r}\alpha(s)ds\\ &\qquad + \int_{\lfloor\frac{z}{r}\rfloor r}^{z}\alpha(s)ds\\ &\quad\geq \int_0^r \alpha(r)ds + \cdots + \int_{(\lfloor\frac{z}{r}\rfloor - 1)r}^{\lfloor\frac{z}{r}\rfloor r}\alpha\left(\left\lfloor\frac{z}{r}\right\rfloor r\right)ds\\ &\qquad + \int_{\lfloor\frac{z}{r}\rfloor r}^{z}\alpha\left(\left\lceil\frac{z}{r}\right\rceil r\right)ds\\ &\quad= \sum_{s=1}^{\lfloor\frac{z}{r}\rfloor}\alpha(sr) + \alpha\left(\left\lceil\frac{z}{r}\right\rceil r\right)\left(z - \left\lfloor\frac{z}{r}\right\rfloor r\right) = w(z).\end{aligned}$$

   Furthermore

$$\begin{aligned}&w(z) - \int_r^z \alpha(s)ds\\ &\quad= \sum_{s=1}^{\lfloor\frac{z}{r}\rfloor}\alpha(sr)r + \alpha\left(\left\lceil\frac{z}{r}\right\rceil r\right)\left(z - \left\lfloor\frac{z}{r}\right\rfloor r\right)\\ &\qquad - \int_r^{\lfloor\frac{z}{r}\rfloor r}\alpha(s)ds - \int_{\lfloor\frac{z}{r}\rfloor r}^{z}\alpha(s)ds\\ &\quad\geq \alpha\left(\left\lfloor\frac{z}{r}\right\rfloor r\right)r + \alpha\left(\left\lceil\frac{z}{r}\right\rceil r\right)\left(z - \left\lfloor\frac{z}{r}\right\rfloor r\right) - \int_{\lfloor\frac{z}{r}\rfloor r}^{z}\alpha(s)ds\\ &\quad\geq \alpha\left(\left\lfloor\frac{z}{r}\right\rfloor r\right)\left(z - \left\lfloor\frac{z}{r}\right\rfloor r\right) \geq 0.\end{aligned}$$

   Since $\lim_{r\to 0}\int_r^z \alpha(s)ds = \int_0^z \alpha(s)ds$, together with squeeze theorem, it follows that $\lim_{r\to 0} w(z) = \int_0^z \alpha(s)ds$ for $z \geq 0$.
5) Under Assumption 2, we have $W(t) = (1/2)\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}} w(\|x_{ij}(t)\|^2)$ for the given r. Suppose that graph $\mathcal{G}(t)$ is disconnected, from Lemma 5, there exist at least $n - 1$ pairs of disconnected vertices in $\mathcal{G}(t)$. Let $\mathcal{S}'$ be the set such that $(i, j) \in \mathcal{S}'$ if and only if $\|x_i(t) - x_j(t)\| \geq R$. Since $w(z) \geq 0$, we have $W(t) \geq (1/2)\sum_{(i,j)\in\mathcal{S}'} w(\|x_{ij}(t)\|^2) = (n-1)w(R^2)$. This conflicts with our assumption that $W(t) < (n-1)w(R^2)$. Therefore, graph $\mathcal{G}(t)$ is connected. ■

The proof of Proposition 2 is based on Lemma 10.

Lemma 10: If the initial states are symmetrically distributed, the states of all agents in model (27) will be symmetrically distributed for any $t \geq 0$.

Proof: Suppose that all the opinions are symmetrically distributed at time $t \geq 0$. For any $i + j = n + 1$, the symmetric distribution implies that $x_i(t) + x_j(t) = x_1(t) + x_n(t)$, and the neighbors of i and j are also symmetrically distributed. That is, for any $k \in \mathcal{N}_i(t)$, there exists a unique $l \in \mathcal{N}_j(t)$, such that $k + l = n + 1$. Moreover, since $x_i(t) + x_j(t) = x_k(t) + x_l(t)$, one has $x_i(t) - x_k(t) = x_l(t) - x_j(t)$, implying that $\alpha_{ik} = \alpha_{jl}$.

Therefore

$$\begin{aligned}(x_i(t+1)+x_j(t+1))-(x_i(t)+x_j(t))\\ =h\sum_{k\in\mathcal{N}_i(t)}\alpha_{ik}(x_k(t)-x_i(t))+h\sum_{l\in\mathcal{N}_j(t)}\alpha_{jl}(x_l(t)-x_j(t))\\ =h\sum_{k\in\mathcal{N}_i(t)}\alpha_{ik}x_k(t)+h\sum_{l\in\mathcal{N}_j(t)}\alpha_{jl}x_l(t)\\ -h\sum_{k\in\mathcal{N}_i(t)}\alpha_{ik}x_i(t)-h\sum_{l\in\mathcal{N}_j(t)}\alpha_{lj}x_j(t)=0\end{aligned}$$

implying that $M=\{x \mid x_i(t)+x_j(t)=x_1(t)+x_n(t),\ i+j=n+1\}$ is a positively invariant set. ■

Proof of Proposition 2: Suppose graph $\mathcal{G}(t)$ has r $(r>1)$ connected components with $\mathcal{V}_1,\ldots,\mathcal{V}_r$ as their vertex sets, and $|\mathcal{V}_i|=n_i$ for $i\in\{1,\ldots,r\}$. Since the agents' states are always symmetrically distributed, we let $n_k=n_j$ for any $k+j=1+r$. Let $g(r)$ be the number of pairs of connected vertices. Then, $g(r)=\sum_{n_i>1}C_{n_i}^2$.

Consider first the case when n is odd. From the symmetry, we have $1\le n_1\le((n-1)/2)$.

If $n_1=1$

$$g(r)\le C^2_{\sum_{n_i>1}n_i}\le C^2_{n-n_1-n_r}=C^2_{n-2}=\frac{n^2-5n+6}{2}.$$

If $n_1=((n-1)/2)$, $g(r)=C^2_{n_1}+C^2_{n_r}=((n^2-4n+3)/4)$.

If $1<n_1<((n-1)/2)$

$$\begin{aligned}g(r)&\le C^2_{n_1}+C^2_{n-n_1-n_r}+C^2_{n_r}\\ &=3n_1^2-2nn_1+\frac{n^2-n}{2}\\ &\le\frac{n^2-8n+27}{4}.\end{aligned}$$

For the case when n is even, from the symmetry, we have $1\le n_1\le(n/2)$.

If $n_1=1$

$$g(r)\le C^2_{\sum_{n_i>1}n_i}\le C^2_{n-n_1-n_r}=C^2_{n-2}=\frac{n^2-5n+6}{2}.$$

If $n_1=(n/2)$, $g(r)=C^2_{n_1}+C^2_{n_r}=(n^2-2n)/4)$.

If $1<n_1<(n/2)$, $g(r)\le C^2_{n_1}+C^2_{n-n_1-n_r}+C^2_{n_r}=3n_1^2-2nn_1+((n^2-n)/2)\le((n^2-6n+12)/4)$.

In conclusion, we can obtain that $g(r)\le((n^2-5n+6)/2)$ for $n\ge 4$. Therefore, the minimal number of pairs of disconnected agents is $f(r)=C_n^2-g(r)\ge 2n-3$. This conflicts with the condition that $f(r)<2n-3$. Hence, graph $\mathcal{G}(t)$ is always connected. ■

REFERENCES

[1] F. Xiao, L. Wang, J. Chen, and Y. Gao, "Finite-time formation control for multi-agent systems," *Automatica*, vol. 45, no. 11, pp. 2605–2611, 2009.

[2] D. V. Dimarogonas and K. H. Johansson, "On the stability of distance-based formation control," in *Proc. 47th IEEE Conf. Decision Control*, Dec. 2008, pp. 1200–1205.

[3] K.-K. Oh and H.-S. Ahn, "Distance-based undirected formations of single-integrator and double-integrator modeled agents in n-dimensional space," *Int. J. Robust Nonlinear Control*, vol. 24, no. 12, pp. 1809–1820, 2014.

[4] X. Dong, B. Yu, Z. Shi, and Y. Zhong, "Time-varying formation control for unmanned aerial vehicles: Theories and applications," *IEEE Trans. Control Syst. Technol.*, vol. 23, no. 1, pp. 340–348, Jan. 2015.

[5] W. Ren, "Consensus strategies for cooperative control of vehicle formations," *IET Control Theory Appl.*, vol. 1, no. 2, pp. 505–512, Mar. 2007.

[6] W. Ren, "Distributed attitude alignment in spacecraft formation flying," *Int. J. Adapt. Control Signal Process.*, vol. 21, nos. 2–3, pp. 95–113, 2007.

[7] G. Jing, Y. Zheng, and L. Wang, "Flocking of multi-agent systems with multiple groups," *Int. J. Control*, vol. 87, no. 12, pp. 2573–2582, 2014.

[8] Y. Jia and L. Wang, "Leader–follower flocking of multiple robotic fish," *IEEE/ASME Trans. Mechatronics*, vol. 20, no. 3, pp. 1372–1383, Jun. 2015.

[9] L. Wang and F. Xiao, "A new approach to consensus problems in discrete-time multiagent systems with time-delays," *Sci. Chin. F, Inf. Sci.*, vol. 50, no. 4, p. 625–635, 2007.

[10] J. Ma, Y. Zheng, and L. Wang, "LQR-based optimal topology of leader-following consensus," *Int. J. Robust Nonlinear Control*, vol. 25, no. 17, pp. 3404–3421, 2015.

[11] J. Lin, A. S. Morse, and B. D. O. Anderson, "The multi-agent rendezvous problem. Part 1: The synchronous case," *SIAM J. Control Optim.*, vol. 46, no. 6, pp. 2096–2119, 2007.

[12] H. Ando, Y. Oasa, I. Suzuki, and M. Yamashita, "Distributed memoryless point convergence algorithm for mobile robots with limited visibility," *IEEE Trans. Robot. Autom.*, vol. 15, no. 5, pp. 818–828, Oct. 1999.

[13] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, Jun. 2003.

[14] U. Krause, "A discrete nonlinear and non-autonomous model of consensus formation," in *Proc. Commun. Differ. Equ.*, 2000, pp. 227–236.

[15] R. Hegselmann and U. Krause, "Opinion dynamics and bounded confidence: Models, analysis and simulation," *J. Artif. Soc. Social Simul.*, vol. 5, no. 3, pp. 1–33, 2002. [Online]. Available: http://jasss.soc.surrey.ac.uk/5/3/2.html

[16] V. D. Blondel, J. M. Hendrickx, and J. N. Tsitsiklis, "On Krause's multi-agent consensus model with state-dependent connectivity," *IEEE Trans. Autom. Control*, vol. 54, no. 11, pp. 2586–2597, Nov. 2009.

[17] F. Ceragioli and P. Frasca, "Continuous and discontinuous opinion dynamics with bounded confidence," *Nonlinear Anal., Real World Appl.*, vol. 13, no. 3, pp. 1239–1251, 2012.

[18] Y. Yang, D. V. Dimarogonas, and X. Hu, "Opinion consensus of modified Hegselmann–Krause models," *Automatica*, vol. 50, no. 2, pp. 622–627, 2014.

[19] J. Ma, Y. Zheng, and L. Wang, "Topology selection for multi-agent systems with opposite leaders," *Syst. Control Lett.*, vol. 93, pp. 43–49, 2016.

[20] M. Andreasson, D. V. Dimarogonas, H. Sandberg, and K. H. Johansson, "Distributed control of networked dynamical systems: Static feedback, integral action and consensus," *IEEE Trans. Autom. Control*, vol. 59, no. 7, pp. 1750–1764, Jul. 2014.

[21] G. Xie and L. Wang, "Consensus control for a class of networks of dynamic agents," *Int. J. Robust Nonlinear Control*, vol. 17, nos. 10–11, pp. 941–959, 2007.

[22] W. Ren and E. Atkins, "Distributed multi-vehicle coordinated control via local information exchange," *Int. J. Robust Nonlinear Control*, vol. 17, nos. 10–11, pp. 1002–1033, Jul. 2007.

[23] L. Wang and F. Xiao, "Finite-time consensus problems for networks of dynamic agents," *IEEE Trans. Autom. Control*, vol. 55, no. 4, pp. 950–955, Apr. 2010.

[24] Y. Zheng, Y. Zhu, and L. Wang, "Consensus of heterogeneous multi-agent systems," *IET Control Theory Appl.*, vol. 5, no. 16, pp. 1881–1888, Nov. 2011.

[25] Y. Zheng and L. Wang, "Distributed consensus of heterogeneous multi-agent systems with fixed and switching topologies," *Int. J. Control*, vol. 85, no. 12, pp. 1967–1976, 2012.

[26] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[27] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Trans. Autom. Control*, vol. 50, no. 2, pp. 169–182, Feb. 2005.

[28] L. Cao, Y. Zheng, and Q. Zhou, "A necessary and sufficient condition for consensus of continuous-time agents over undirected time-varying networks," *IEEE Trans. Autom. Control*, vol. 56, no. 8, pp. 1915–1920, Aug. 2011.

[29] J. M. Hendrickx and J. N. Tsitsiklis, "Convergence of type-symmetric and cut-balanced consensus seeking systems," *IEEE Trans. Autom. Control*, vol. 58, no. 1, pp. 214–218, Jan. 2013.

[30] J. Qin, H. Gao, and W. X. Zheng, "Exponential synchronization of complex networks of linear systems and nonlinear oscillators: A unified analysis," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 26, no. 3, pp. 510–521, Mar. 2015.

[31] H.-T. Zhang and Z. Chen, "Consensus acceleration in a class of predictive networks," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 25, no. 10, pp. 1921–1927, Oct. 2014.

[32] Y. Zheng and L. Wang, "Consensus of switched multiagent systems," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 63, no. 3, pp. 314–318, Mar. 2016.

[33] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, "Novel type of phase transition in a system of self-driven particles," *Phys. Rev. Lett.*, vol. 75, no. 6, pp. 1226–1229, Aug. 1995.

[34] F. Cucker and J.-G. Dong, "Avoiding collisions in flocks," *IEEE Trans. Autom. Control*, vol. 55, no. 5, pp. 1238–1243, May 2010.

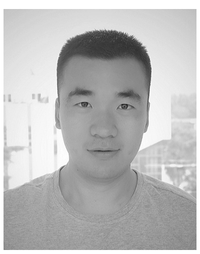
[35] J. Cortés, S. Martínez, and F. Bullo, "Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions," *IEEE Trans. Autom. Control*, vol. 51, no. 8, pp. 1289–1298, Aug. 2004.

[36] P. Lin and W. Ren, "Distributed shortest distance consensus problem in multi-agent systems," in *Proc. 51st IEEE Conf. Decision Control*, Dec. 2012, pp. 4696–4701.

[37] T. Li, F. Wu, and J.-F. Zhang, "Multi-agent consensus with relative-state-dependent measurement noises," *IEEE Trans. Autom. Control*, vol. 59, no. 9, pp. 2463–2468, Sep. 2014.

[38] J. Cortés, "Finite-time convergent gradient flows with applications to network consensus," *Automatica*, vol. 42, no. 11, pp. 1993–2000, 2006.

[39] D. B. West, *Introduction to Graph Theory*. Upper Saddle River, NJ, USA: Prentice-Hall, 2001.



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