Distributed algorithms for solving a class of convex feasibility problems

Kaihong Lu, Gangshan Jing, and Long Wang

Abstract

In this paper, a class of convex feasibility problems (CFPs) are studied for multi-agent systems through local interactions. The objective is to search a feasible solution to the convex inequalities with some set constraints in a distributed manner. The distributed control algorithms, involving subgradient and projection, are proposed for both continuous- and discrete-time systems, respectively. Conditions associated with connectivity of the directed communication graph are given to ensure convergence of the algorithms. It is shown that under mild conditions, the states of all agents reach consensus asymptotically and the consensus state is located in the solution set of the CFP. Simulation examples are presented to demonstrate the effectiveness of the theoretical results.

Index Terms

Multi-agent systems; Consensus; Convex inequalities; Subgradient; Projection.

I. INTRODUCTION

Distributed coordination control of multi-agent systems (MASs) has been intensively investigated in various areas including engineering, natural science, and social science [1]-[3]. As a fundamental coordination problem, the consensus which requires that a group of autonomous agents achieve a common state has attracted much attention, see [4]-[11]. This is due to its

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wide applications in distributed control and estimation [12], distributed optimization [13]-[15] and distributed methods for solving linear equations [16], [17].

Researches on consensus can be roughly categorized depending on whether the agents have continuous- or discrete- time dynamics. Noticeable works focusing on the multi-agent systems include [6], [9], [18], [19] for the continuous-time case and [5], [19]-[21] for the discrete-time case. In the aforementioned works, the agents interact with each other through a network and each agent adjusts its own state by using only local information from its neighbors. Within this framework, connectivity of the communication graph plays a key role in achieving consensus, and consequently several conditions of the connectivity have been established. For example, the communication graph must have a spanning tree when the topology is fixed [6], while the union of the communication graphs should have a spanning tree frequently enough as the system evolves when the topology is switching [9], [21]. In addition, infinitely-joint connectedness, i.e., the infinitely occurring communication graphs are jointly connected, is necessary to make the agents reach consensus when the topology is time-varying [18], [19].

In recent years, the constrained consensus problem that seeks to reach state agreement in the intersection of a number of convex sets has been widely investigated. In [22], a projection-based consensus algorithm was proposed when the communication graph is balanced. This algorithm with time delays was studied in [24], where the union of the communication graphs within a period was assumed to be strongly connected. The problem was extended to the continuous-time case in [25], where each set serves as an optimal solution set of a local objective function, and the global optimal solution is achieved as long as the intersection of the constrained sets is computed. By taking the advantages of the property that the solution set of linear equations is an affine set, the projection-based consensus algorithm in [25] was successfully applied to solving linear equations in [26], where the projection operator in [25] was replaced with a special affine projection operator. Unlike the distributed algorithm for solving linear equations in [16], the projection-based consensus algorithm in [26] does not need to restrict each agent's initial state within the solution set of its corresponding equations. The methods in [22]-[26] are useful for the computation of the intersection when the projections onto the local sets are easily calculated. However, in general, the application of the projected method usually requires the solution of an auxiliary minimization problem associated with the projection onto the local set at each time. This might lead to a limitation on its applications.

Comparing with computing the intersection and solving linear equations, a more general problem is solving CFPs, which usually needs to solve linear equations and convex inequalities simultaneously, and ensure the solution to be in the intersection of some simple convex sets. Applications of solving CFPs arise in different fields, such as pattern recognition [27], signal processing [28] and image restoration [29], [30]. It is also well known that some convex programming problems can be transformed into an equivalent CFP through the Karush-Kuhn-Tucker condition [31]. For example, the linear program problem in [32] can be transformed into a set of linear equations and inequalities. Inspired by the distributed methods for solving linear equations [16], [26], distribute methods for CFPs will be studied in this paper. Different from linear equations, the solution set of a CFP is usually not a simple affine set due to the existence of inequalities which can even be nonlinear, thus it is necessary to develop alternative methods for solving this problem.

In this paper, distributed algorithms, involving subgradient and projection, are proposed for multi-agent systems to solve the CFP involving convex inequalities. Here the distributed control algorithms are designed for the continuous- and the discrete-time systems, respectively. Our aim is to obtain the graphic criteria for the convergence of these algorithms. One of the challenge is that, the subgradient and projection operations lead to nonlinearity of the algorithms. To deal with this problem, the control inputs are decomposed into a linear part involving the traditional consensus term and a nonlinear part involving the subgradient and projection operations. The linear part is analyzed by using the graph theory and some basic theories of stability associated with linear systems, while the nonlinear part is done by Lyapunov theory. The contributions of this paper are summarized as follows:

(1) Both continuous- and discrete-time distributed algorithms are provided for solving CFPs. Different from the distributed algorithms for solving linear equations in [16], [17], in which the algorithms need to restrict each agent's initial state within the solution set of its corresponding equations, the CFPs can be solved by the presented algorithms under arbitrary initial states.

(2) The continuous-time distributed gradient-based algorithm has also been investigated in [36], where convergence of the algorithm relies on a time-varying parameter. Our algorithm does not involve a time-varying parameter and it does not require the assumption on boundedness of the subgradient as in [36]. We prove that, if the directed graph is fixed and strongly connected, all agents' states will reach a common point asymptotically and the point is located in the solution

set of the CFP. Moreover, we find that the CFP can be solved if the δ -graph associated with a time-varying graph is strongly connected.

(3) Discrete-time distributed subgradient-based algorithms have been studied in [22], where the communication graph is balanced. Unlike [22], [23], in our algorithm, only relative information between the agents is required and the convergence can also be ensured when the communication graph is unbalanced. We prove that the effectiveness of the presented algorithm can be guaranteed when the directed graph is strongly connected.

This paper is organized as follows. In Section II, we present some notions in graph theory and state the problem studied in this paper. In Section III, centralized algorithms in both continuousand discrete-time cases for the CFP are focused on and the convergence of them is analyzed. In Section IV, the distributed control algorithm in continuous-time case is presented for the MAS to solve the CFP, and the convergence is analyzed under both fixed and time-varying communication graphs. The discrete-time case is studied in Section V. In Section VI, a distributed gradient-based algorithm is designed for a CFP involving linear inequalities. Simulation examples are presented in Section VII. Section VIII concludes the whole paper.

Notation: Throughout this paper, we use |a| to represent the absolute value of scalar a. \mathbb{R} and \mathbb{C} denote the set of real number and the set of complex number, respectively. Let \mathbb{R}^m be the *m*-dimensional real vector space and \mathbb{C}^m be the complex one. For a given vector $x \in \mathbb{R}^m$, $x > 0 (\geq 0)$ implies that each entry of vector x is greater than (not less than) zero. ||x|| denotes the standard Euclidean norm, i.e., $||x|| = \sqrt{x^T x}$. For a function $g(\cdot) : \mathbb{R}^m \to \mathbb{R}$, we denote its plus function by $g^+(\cdot) = \max[g(\cdot), 0]$. $\mathbf{1}_n$ denotes the *n*-dimensional vector with elements being all ones. I_n denotes the $n \times n$ identity matrix. The transposes of matrix A and vector x are denoted as A^T and x^T , respectively. For any two vectors u and v, the operator $\langle u, v \rangle$ denotes the inner product of u and v. For matrices A and B, the Kronecker product is denoted by $A \otimes B$.

II. PRELIMINARY AND PROBLEM FORMULATION

A. Graph theory

The communication topology is denoted by $\mathcal{G}(\mathcal{A}(t)) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t)), \mathcal{V}$ is a set of vertices, $\mathcal{E}(t) \subset \mathcal{V} \times \mathcal{V}$ is an edge set, and the weighted matrix $\mathcal{A}(t) = (a_{ij}(t))_{n \times n}$ is a non-negative matrix for adjacency weights of edges. If node *i* can receive the information from node *j*, then node *j* is called as node *i*'s neighbor and it is denoted by $(j, i) \in \mathcal{E}(t)$ and $a_{ij}(t) > 0$. Otherwise, $a_{ij}(t) = 0$. Denote $N_i(t) = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}(t)\}$ to represent the neighbor set of node *i* at time *t*. The Laplacian matrix of the graph is defined as $L(t) = (l_{ij}(t))_{n \times n}$, where $l_{ij}(t) = -a_{ij}(t)$ if $i \neq j$ and $l_{ij}(t) = \sum_{j=1}^{n} a_{ij}(t)$ if i = j for any $i = 1, \dots, n$. For a fixed and directed graph $\mathcal{G}(\mathcal{A})$, a path of length *r* from node i_1 to node i_{r+1} is a sequence of r + 1 distinct nodes $i_1 \dots, i_{r+1}$ such that $(i_q, i_{q+1}) \in \mathcal{E}$ for $q = 1, \dots, r$. If there exists a path between any two nodes in \mathcal{V} , then $\mathcal{G}(\mathcal{A})$ is said to be strongly connected. A directed graph, where every node has exactly one neighbor except the root, is said to be a directed tree. A spanning tree of a directed graph is a directed tree formed by graph edges that connect all the nodes of the graph [33]. We say that a graph has a spanning tree if a subset of the edges forms a spanning tree.

For a time-varying and directed graph $\mathcal{G}(\mathcal{A}(t))$, (j, i) is called a δ -edge if there always exist two positive constants T and δ such that $\int_t^{t+T} a_{ij}(s)ds \ge \delta$ for any $t \ge 0$. A δ -graph, induced by $\mathcal{G}(\mathcal{A}(t))$, is defined as $\mathcal{G}_{(\delta,T)} = (\mathcal{V}, \mathcal{E}_{(\delta,T)})$, where $\mathcal{E}_{(\delta,T)} = \{(j,i) \in \mathcal{V} \times \mathcal{V} | \int_t^{t+T} a_{ij}(s)ds \ge \delta$ for any $t \ge 0\}$. The communication graph $\mathcal{G}(\mathcal{A}(t))$ is said to be balanced if the sum of the interaction weights from and to an agent i are equal, i.e., $\sum_{j=1}^n a_{ij}(t) = \sum_{j=1}^n a_{ji}(t)$.

Lemma 1: [5] For a fixed graph $\mathcal{G}(\mathcal{A})$, if $\mathcal{G}(\mathcal{A})$ has a spanning tree, then the Laplacian matrix L has one simple 0 eigenvalue and the other eigenvalues have positive real parts.

Lemma 2: [6] For a fixed graph $\mathcal{G}(\mathcal{A})$, if $\mathcal{G}(\mathcal{A})$ is strongly connected, then there exists a vector $w = [w_1 \cdots w_n]^T > 0$ such that $w^T L = 0$.

For ease of description, if $\mathcal{G}(\mathcal{A})$ has a spanning tree, we use $\lambda_1(L)$ to represent the 0 eigenvalue and $\lambda_i(L), i = 2, \dots, n$ to represent other non-zero eigenvalues.

B. Convex analysis

A function $f(\cdot) : \mathbb{R}^m \to \mathbb{R}$ is convex if it holds $f(\gamma x + (1 - \gamma)y) \leq \gamma f(x) + (1 - \gamma)f(y)$ for any $x \neq y \in \mathbb{R}^m$ and $0 < \gamma < 1$. For convex function f(x), if $\langle \nabla f(x), y - x \rangle \leq f(y) - f(x)$ holds for any $y \in \mathbb{R}^m$, then $\nabla f(x)$ is a subgradient of function f at point $x \in \mathbb{R}^m$. There must exist subgradients for any convex function. Furthermore, if the convex function is differentiable, its gradient is the unique subgradient.

Given a set $\Omega \subset \mathbb{R}^m$, it is called as a convex set if $\gamma x + (1-\gamma)y \in \Omega$ for any scalar $0 < \gamma < 1$ and $x, y \in \Omega$. For a closed convex set Ω , let $||x||_{\Omega} \stackrel{\Delta}{=} \inf_{y \in \Omega} ||x - y||$ denote the standard Euclidean distance of vector $x \in \mathbb{R}^m$ from Ω . Then, there is a unique element $P_{\Omega}(x) \in \Omega$ such that $||x - P_{\Omega}(x)|| = ||x||_{\Omega}$, where $P_{\Omega}(\cdot)$ is called the projection onto the set Ω [34]. Moreover, $P_{\Omega}(\cdot)$ has the non-expansiveness property: $||P_{\Omega}(x) - P_{\Omega}(y)|| \le ||x - y||$ for any $x, y \in \mathbb{R}^m$.

Lemma 3: For a convex function $g(\cdot) : \mathbb{R}^m \to \mathbb{R}$, suppose the set $X = \{x \in \mathbb{R}^m | g^+(x) = 0\}$ is non-empty, it holds $z \in X$ if and only if 0 is a subgradient of the plus function g^+ at point z.

Proof: Sufficiency. By the definition of $g^+(\cdot)$, we know function $g^+(\cdot)$ is convex. Therefore, the subgradient of function $g^+(\cdot)$ always exists. If 0 is a subgradient of the plus function g^+ at point z, by the definition of the subgradient, we have $g^+(y) - g^+(z) \ge 0^T(y-z) = 0$ for any $y \in \mathbb{R}^m$. Let $y \in X$, then we have $-g^+(z) \ge 0$. By this and the fact that $g^+(z) \ge 0$, it can be concluded that $g^+(z) = 0$.

Necessity. If $z \in X$, we have $g^+(z) = 0$. Due to the fact that $g^+(y) \ge 0$, we have $g^+(y) - 0 \ge 0^T(y-z)$ for any $y \in \mathbb{R}^m$. Thus, 0 is a subgradient of the plus function g^+ at point z.

Lemma 4: [22] Given a closed convex set $\Omega \subset \mathbb{R}^m$, it holds

$$\langle P_{\Omega}(x) - x, x - y \rangle \leq - \|x\|_{\Omega}^2$$

for any $x \in \mathbb{R}^m, y \in \Omega$.

C. Problem formulation

Consider a MAS consisting of n agents, labeled by set $\mathcal{V} = \{1, \dots, n\}$. Here we consider agents with both continuous-time dynamics

$$\dot{x}_i(t) = u_i(t), i \in \mathcal{V} \tag{1}$$

and discrete-time dynamics

$$x_i(t+1) = x_i(t) + u_i(t), i \in \mathcal{V}$$

$$\tag{2}$$

where $x_i(t) \in \mathbb{R}^m$ and $u_i(t) \in \mathbb{R}^m$ are respectively, the state and input of agent *i*. The objectives of this paper are to design $u_i(t)$ for (1) and (2) by using only local information to solve the following CFP:

$$\begin{cases} g_i(x) \le 0 \\ x \in X : \stackrel{\Delta}{=} \cap_{i=1}^n X_i \end{cases} \qquad i = 1, \cdots, n$$
(3)

where $x \in \mathbb{R}^m$, $g_i(\cdot) : \mathbb{R}^m \to \mathbb{R}$ is a convex function, it is continuous on $(-\infty, \infty)$. Each X_i is a closed convex set. Agent *i* can only have access to the information associated with

subgradient $\nabla g_i^+(\cdot)$ and projection $P_{X_i}(\cdot)$. We assume each $\nabla g_i^+(\cdot)$ is piecewise continuous for any $i = 1, \dots, n$.

Remark 1: Note that if and only if $x \in X_i$, it holds $x = P_{X_i}(x)$. If $x = P_{X_i}(x)$ for all $i = 1, \dots, n$, then x belongs to their intersection. Since the algorithms in the following sections refer to the projection operator $P_{X_i}(\cdot)$, here we only consider some convex sets X_i onto which the projection $P_{X_i}(x)$ can be easily calculated or their expressions could be given in detail at any point x. For example, if set X represents the solution set of linear equation $a^T x - b = 0$, i.e., $X = \{x | a^T x - b = 0\}$, where $a, x \in \mathbb{R}^m, b \in \mathbb{R}$, it is easy to show that $P_X(x) = \left(I - \frac{aa^T}{\|a\|^2}\right)x + \frac{ba}{\|a\|^2}$ is a projection of x onto set X. Consequently, it is not difficult to find that the algorithms in the following sections are also available to the CFP involving linear equations.

The solution set of CFP (3) is denoted by X^* and the following assumption is adopted throughout the paper.

Assumption 1: \mathbf{X}^* is non-empty.

Note that a vector x^* belongs to \mathbf{X}^* , if and only if it holds that $x^* \in X$ and $g_i^+(x^*) = 0$ for each $i \in \{1, \dots, n\}$.

III. CENTRALIZED ALGORITHMS FOR CFPS

In this section, we focus on the following CFP

$$\begin{cases} g(x) \le 0\\ x \in X \end{cases}$$
(4)

where $x \in \mathbb{R}^m$, $g(\cdot) : \mathbb{R}^m \to \mathbb{R}$ is a convex function, and X is a closed convex set.

A. Continuous-time case

To solve CFP (4), the following continuous-time subgradient and projection-based algorithm is proposed.

$$\dot{x}(t) = -\alpha(t)[x(t) - P_X(x(t))] - \beta(t)\nabla g^+(x(t))$$
(5)

where $\alpha(t), \beta(t) \in \mathbb{R}$.

Theorem 1: Suppose CFP (4) has a non-empty solution set \mathbf{X}^* , if $\alpha(t) \ge 0$ and $\beta(t) \ge 0$ satisfy that $\int_0^\infty \alpha(t) \to \infty$ and $\int_0^\infty \beta(t) \to \infty$, then x(t) in (5) converges to a vector x^* in set \mathbf{X}^* .

Proof: Define a positive-definite Lyapunov function candidate $V(t) = \frac{1}{2} ||x(t) - x_0||^2$, where $x_0 \in \mathbf{X}^*$. By the definition of g^+ , it holds $g^+(x_0) = ||x_0||_X = 0$. Based on the property of the subgradient, we have $\langle x(t) - x_0, \nabla g^+(x(t)) \rangle \ge g^+(x(t))$. Taking the derivative of function V(t) with respect to t yields

$$\dot{V}(t) = \langle x(t) - x_0, \dot{x}(t) \rangle$$

$$= \langle x(t) - x_0, -\alpha(t)[x(t) - P_X(x(t))] - \beta(t)\nabla g^+(x(t)) \rangle$$

$$= -\alpha(t) \langle x(t) - x_0, x(t) - P_X(x(t)) \rangle - \beta(t) \langle x(t) - x_0, \nabla g^+(x(t)) \rangle$$

$$\leq -\alpha(t) \langle x(t) - x_0, x(t) - P_X(x(t)) \rangle - \beta(t)g^+(x(t)).$$
(6)

By Lemma 4, we know $-\langle x(t) - x_0, x(t) - P_X(x(t)) \rangle \leq -||x(t)||_X^2 \leq 0$. Note that $g^+(x(t)) \geq 0$. Thus, $\dot{V}(t) \leq 0$. Moreover, V(t) is bounded by zero, it can be concluded that V(t) converges and $V(\infty)$ exists, which implies $||x(t) - x_0||$ converges. By inequality (6), we have

$$\int_{0}^{\infty} \alpha(t) \|x(t)\|_{X}^{2} d_{t} + \int_{0}^{\infty} \beta(t) g^{+}(x(t)) d_{t} \leq V(0) - V(\infty) < \infty.$$

Since $\alpha(t) \|x(t)\|_X^2$ and $\beta(t)g^+(x(t))$ are both non-negative, then we have $\int_0^\infty \alpha(t) \|x(t)\|_X^2$ $d_t < \infty$ and $\int_0^\infty \beta(t)g^+(x(t))d_t < \infty$. These and the facts $\int_0^\infty \alpha(t) \to \infty$ and $\int_0^\infty \beta(t) \to \infty$ imply $\lim_{t\to\infty} \inf \|x(t) - P_X(x(t))\| = \lim_{t\to\infty} \inf g^+(x(t)) = 0$. Thus, there exists a subsequence $\{x(t_k)\}$ of x(t) such that $\lim_{k\to\infty} x(t_k) = \lim_{t\to\infty} \inf x(t) = x^*$, where x^* is a point in the solution set of CFP (4). Moreover, note that V(x(t)) converges, it can be concluded that $\lim_{t\to\infty} x(t) = x^* \in \mathbf{X}$. Hence, the validity of the result is verified.

Corollary 1: Suppose CFP (4) has a non-empty solution set \mathbf{X}^* , if x(t) adjusts its value with the following dynamics

$$\dot{x}(t) = -[x(t) - P_X(x(t))] - \nabla g^+(x(t))$$

then x(t) converges to a vector x^* in set \mathbf{X}^* .

B. Discrete-time case

Now we present the discrete-time algorithm for CFP (4).

$$\begin{cases} \xi(t) = x(t) - \beta(t)\nabla g^+(x(t)) \\ \varphi(t) = \alpha(t)\left(\xi(t) - P_X(\xi(t))\right) \\ x(t+1) = \xi(t) - \varphi(t) \end{cases}$$
(7)

where $P_X(\cdot)$ and $\nabla g^+(x(t))$ are defined as those in (5).

Assumption 2: $\nabla g^+(x(t)) \leq K$ for some $K \geq 0$.

Lemma 5: [35] Let $\{z(t)\}$ be a non-negative scalar sequence such that

$$z(t+1) \le (1+a(t))z(t) - b(t) + c(t)$$

for all $t \ge 0$, if $a(t) \ge 0, b(t) \ge 0, c(t) \ge 0$ with $\sum_{t=0}^{\infty} a(t) < \infty$ and $\sum_{t=0}^{\infty} c(t) < \infty$, then the sequence $\{z(t)\}$ converges to some constant z^* and $\sum_{t=0}^{\infty} b(t) < \infty$.

Theorem 2: Under Assumptions 2, if CFP (4) has a non-empty solution set \mathbf{X}^* , and $\alpha(t)$, $\beta(t)$ satisfy

(a)
$$\alpha(t) \in [0, 1]$$
, $\sum_{t=0}^{\infty} \alpha(t) \to \infty$ and $\sum_{t=0}^{\infty} \alpha^2(t) < \infty$;
(b) $0 \le \beta(t) \le \infty$, $\sum_{t=0}^{\infty} \beta(t) \to \infty$ and $\sum_{t=0}^{\infty} \beta^2(t) < \infty$.
Then, $x(t)$ in (7) converges to a vector x^* in set \mathbf{X}^* .

Proof: We choose the Lyapunov function candidate as $V(t) = ||x(t) - x_0||^2$, where $x_0 \in \mathbf{X}^*$. Taking the difference of function V(t) along with (7) yields

$$\Delta V(t) = V(t+1) - V(t)$$

$$= \|\xi(t) - \varphi(t) - x_0\|^2 - \|x(t) - x_0\|^2$$

$$= \|(1 - \alpha(t))(\xi(t) - x_0) + \alpha(t)(P_X(\xi(t)) - x_0)\|^2 - \|x(t) - x_0\|^2$$

$$\leq \left((1 - \alpha(t))\|\xi(t) - x_0\| + \alpha(t)\|P_X(\xi(t)) - x_0\|\right)^2 - \|x(t) - x_0\|^2$$

$$\leq \|\xi(t) - x_0\|^2 - \|x(t) - x_0\|^2$$
(8)

where the last inequality follows from the non-expansiveness property of projection operator, i.e., $||P_X(\xi(t)) - x_0|| \le ||\xi(t) - x_0||$. Moreover, we have

$$\|\xi(t) - x_0\|^2 \le \|x(t) - x_0\|^2 - 2\beta(t) \langle \nabla g^+(x(t)), x(t) - x_0 \rangle + \beta^2(t) K \le \|x(t) - x_0\|^2 - 2\beta(t) (g^+(x(t)) - g^+(x_0)) + \beta^2(t) K.$$
(9)

From inequalities (8) and (9), we have $\Delta V(t) \leq \beta^2(t)K$. Thus, it holds that $V(t) \leq V(0) + \sum_{t=0}^{t-1} \beta^2(t)K \leq V(0) + \sum_{t=0}^{\infty} \beta^2(t)K < \infty$. By the definition of V(t), it can be concluded that x(t) is bounded. Since $\|\beta(t)\nabla g^+(x(t))\| < \infty$, $\xi(t)$ is bounded. This and the continuity of

 $P_X(\xi(t))$ imply $\|\xi(t) - P_X(\xi(t))\| < \infty$. Denote $\nabla(t) = \beta(t)\nabla g^+(x(t))$, since $\sum_{t=0}^{\infty} \alpha^2(t) < \infty$ and $\sum_{t=0}^{\infty} \beta^2(t) < \infty$, it can be concluded that $\sum_{t=0}^{\infty} \|\nabla(t)\|^2 < \infty$ and $\sum_{t=0}^{\infty} \|\varphi(t)\|^2 < \infty$. Similar to (8), we also have

$$\begin{aligned} \Delta V(t) &= V(t+1) - V(t) \\ &= -2\langle \nabla(t) + \varphi(t), x(t) - x_0 \rangle + \|\nabla(t) + \varphi(t)\|^2 \\ &= -2\langle \nabla(t), x(t) - x_0 \rangle - 2\langle \varphi(t), \xi(t) - x_0 \rangle \\ &- 2\langle \varphi(t), \nabla(t) \rangle + \|\nabla(t) + \varphi(t)\|^2 \\ &= -2\langle \nabla(t), x(t) - x_0 \rangle - 2\langle \varphi(t), \xi(t) - x_0 \rangle + \|\nabla(t)\|^2 + \|\varphi(t)\|^2 \\ &\leq -2\beta(t)g^+(x(t)) - 2\alpha(t)\|\xi(t)\|_X^2 + \|\nabla(t)\|^2 + \|\varphi(t)\|^2 \\ &= -2\beta(t)g^+(x(t)) - 2\alpha(t)\|\xi(t)\|_X^2 + \|\nabla(t)\|^2 + \|\varphi(t)\|^2. \end{aligned}$$

Recall the fact that $\sum_{t=0}^{\infty} \|\nabla(t)\|^2 + \|\varphi(t)\|^2 < \infty$ and $-2\beta(t)g^+(x(t)) - 2\alpha(t)\|\xi(t)\|_X^2 < 0$, by Lemma 5, it can be concluded $\|x(t) - x_0\|$ converges and it holds $\sum_{t=0}^{\infty} \left(\beta(t)g^+(x(t)) + \alpha(t)\|\xi(t)\|_X^2\right) < \infty$. Since $\beta(t)g^+(x(t)) > 0$ and $\alpha(t)\|\xi(t)\|_X^2 > 0$ for all t > 0, we have $\sum_{t=0}^{\infty} \beta(t)g^+(x(t)) < \infty$ and $\sum_{t=0}^{\infty} \alpha(t)\|\xi(t)\|_X^2 < \infty$. By the facts $\sum_{t=0}^{\infty} \alpha(t) \to \infty$ and $\sum_{t=0}^{\infty} \beta(t) \to \infty$, we have $\lim_{t\to\infty} \inf \|\xi(t) - P_X(\xi(t))\| = \lim_{t\to\infty} \inf g^+(x(t)) = 0$. Thus, there exists a subsequence $\{x(t_k)\}$ of x(t) such that $\lim_{k\to\infty} x(t_k) = x^*$, where x^* is a vector such that $g^+(x^*) = 0$. By the fact $\|x(t) - x_0\|$ converges, we can conclude $\lim_{t\to\infty} x(t) = x^*$. Furthermore, note that $\nabla(t) \to 0$ as $t \to \infty$, thus $\lim_{t\to\infty} \inf \|\xi(t) - P_X(\xi(t))\| = 0$ and $\lim_{t\to\infty} x(t) = x^*$ imply $\lim_{t\to\infty} \|x^* - P_X(x^*)\| = 0$. Therefore, x^* is a solution to CFP (4), i.e., $x^* \in \mathbf{X}^*$.

IV. CONTINUOUS-TIME DISTRIBUTED CONTROL ALGORITHMS FOR SOLVING CFPs

In this section, we focus on solving CFP (3) for continuous-time MAS (1) in a distributed manner, which means that each agent has access to only its own state and that from its neighbors. The following input is proposed.

$$\begin{cases} u_{i}(t) = \sum_{i \in N_{i}(t)} a_{ij}(t)(x_{j}(t) - x_{i}(t)) + \phi_{i}(t) & i \in \mathcal{V} \\ \phi_{i}(t) = -\tau \left([x_{i}(t) - P_{X_{i}}(x_{i}(t))] + \nabla g_{i}^{+}(x_{i}(t)) \right) & (11) \end{cases}$$

where τ is a positive coefficient. Note that ϕ_i depends on only agent *i*'s own state, so (11) is distributed. Based on Lemma 3 in Section II, here we set $\nabla g_i^+(x) = 0$ if $g_i(x) \leq 0$ and $\nabla g_i^+(x) = \nabla g_i(x)$ otherwise.

Remark 2: If we set $\tau = 0$ in algorithm (11), then it will become a typical linear consensus algorithm for MASs studied in [5], [6]. In this case, MASs reach consensus asymptotically if the communication graph is fixed and has a spanning tree. The distributed subgradient-based algorithm was studied for continuous-time multi-agent systems to optimize a sum of convex objective functions in [36], but the convergence of the algorithm relies on a time-varying parameter and the projection term was not involved.

Let $x(t) = [x_1^T(t), \dots, x_n^T(t)]^T$ and $\phi(t) = [\phi_1^T(t), \dots, \phi_n^T(t)]^T$, MAS (1) with (11) can be rewritten as

$$\dot{x}(t) = -\left(L(t) \otimes I_m\right) x(t) + \phi(t). \tag{12}$$

Lemma 6: [37] Let b(t) be a bounded function, if $\lim_{t\to\infty} b(t) = b$ and $0 < \gamma < 1$, then $\lim_{t\to\infty} \int_0^t \gamma^{t-s} b(s) \, ds = -\frac{b}{\ln\gamma}$.

Lemma 7: [38] Given a symmetric matrix $P = (p_{ij})_{n \times n}$ with 0 eigenvalue and a vector $x = [x_1, \dots, x_n]^T$, if $P\mathbf{1}_n = 0$, then it holds $x^T P x = -\sum_{i=1}^n \sum_{j=i+1}^n p_{ij}(x_i - x_j)^2$.

Lemma 8: Given a linear system $\dot{x}(t) = Ax(t) + u(t)$, if the state matrix $A \in \mathbb{R}^{n \times n}$ is Hurwitz stable and $u(t) \in \mathbb{R}^n$ satisfies $||u(t)|| < \infty$ and $\lim_{t \to \infty} u(t) = 0$, then the linear system is asymptotically stable to zero, i.e., $\lim_{t \to \infty} x(t) = 0$.

Proof: Since matrix A is Hurwitz stable, all of its eigenvalues have negative real parts. Based on theory of Schur's unitary triangularization, there exists a unitary matrix $U \in \mathbb{C}^{n \times n}$ such that

$$U^{H}AU = \begin{bmatrix} \lambda_{1} & \lambda_{12} & \cdots & \lambda_{1n} \\ 0 & \lambda_{2} & \lambda_{23} & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_{n} \end{bmatrix} \stackrel{\Delta}{=} \Lambda$$

where λ_i is the eigenvalue of matrix A, $i = 1, \dots, n$; U^H is the conjugate transpose matrix of U. Denote $y(t) = U^H x(t)$ and $r(t) = U^H u(t)$, we have $\dot{y}(t) = \Lambda y(t) + r(t)$. By the fact that $\lim_{t \to \infty} u(t) = 0$, we have $\lim_{t \to \infty} r(t) = 0$. Let $y(t) = [y_1(t), \dots, y_n(t)]^T$ and r(t) = $[r_1(t), \cdots, r_n(t)]^T, \text{ we have } \dot{y}_n(t) = \lambda_n y_n(t) + r_n(t). \text{ The term } r_n(t) \text{ can be viewed as an control input of the linear system and we have } y_n(t) = e^{\lambda_n t} y_n(0) + \int_0^t e^{\lambda_n (t-\tau)} r_n(\tau) d_{\tau}. \text{ Since the real part of } \lambda_n \text{ is negative, it holds } 0 < e^{\lambda_n} < 1. \text{ By Lemma 6, it can be concluded that } \lim_{t \to \infty} y_n(t) = 0.$ Since $\dot{y}_i(t) = \lambda_i y_i(t) + \left(\sum_{j=1}^n \lambda_{i(i+j)} y_{i+j}(t) + r_i(t)\right).$ Through the similar approach for $y_n(t)$, we can conclude $\lim_{t \to \infty} \left(\sum_{j=1}^n \lambda_{i(i+j)} y_{i+j}(t) + r_i(t)\right) = 0.$ Reusing Lemma 6 yields $\lim_{t \to \infty} y_i(t) = 0$ for any $i = 1, \cdots, n$. This and the fact x(t) = Uy(t) imply $\lim_{t \to \infty} x(t) = 0.$

To prove the fact that MAS (1) with (11) solves CFP (3), it is necessary to analyze the convergence of MAS (1) with (11). Obviously, the conditions for convergence depend on the connectivity of the graphs. In the following, we will provide the convergence conditions under the fixed graph and the time-varying graph, respectively.

A. Convergence under the fixed communication graph

Proposition 1: Suppose $\|\phi_i(t)\| < \infty$ and $\lim_{t\to\infty} \phi_i(t) = 0$ in (11), $i \in \mathcal{V}$, if the fixed graph $\mathcal{G}(\mathcal{A})$ is directed and has a spanning tree, then MAS (1) with (11) reaches consensus asymptotically.

Proof: Define a variable
$$\hat{x}(t) = \sum_{i=1}^{n} \frac{w_i x_i(t)}{\sum_{i=1}^{n} w_i} = \left(\frac{w^T}{\mathbf{1}^T w} \otimes I_m\right) x(t)$$
, where $w = [w_1 \cdots w_n]^T$ is

L's left eigenvector associated with 0 eigenvalue. Based on (12), we have $\dot{\hat{x}}(t) = \frac{\left(w^T \otimes I_m\right)}{\mathbf{1}^T w} u(t)$. Denote $e_i(t) = x_i(t) - \hat{x}(t)$ and $e(t) = \left[e_1^T(t), \cdots, e_n^T(t)\right]^T$. Note that if $\lim_{t \to \infty} e(t) = 0$, then MAS (1) with (11) reaches consensus. From (12), we have

$$\dot{e}(t) = -(L \otimes I_m)x(t) + \left(\left(I_n - \frac{\mathbf{1}_n w^T}{\mathbf{1}_n^T w}\right) \otimes I_m\right)\phi(t)$$

$$= -(L \otimes I_m)x(t) + (L \otimes I_m)\left(\frac{\mathbf{1}_n w^T}{\mathbf{1}_n^T w} \otimes I_m\right)x(t)$$

$$+ \left(\left(I_n - \frac{\mathbf{1}_n w^T}{\mathbf{1}_n^T w}\right) \otimes I_m\right)\phi(t)$$

$$= -(L \otimes I_m)e(t) + \left(\left(I_n - \frac{\mathbf{1}_n w^T}{\mathbf{1}_n^T w}\right) \otimes I_m\right)\phi(t)$$
(13)

where the second equation holds for the fact that $L\mathbf{1}_n = 0$. Note that $\frac{1}{\sqrt{w^T w}}L^T w = 0$. Now we use $\frac{1}{\sqrt{w^T w}}w$ to form a set of orthonormal basis on $\in \mathbb{C}^n$, denoted by $\frac{1}{\sqrt{w^T w}}w, p_2, \cdots, p_n$. We define $P = (\frac{1}{\sqrt{w^T w}}w, p_2, \cdots, p_n)$. It is obvious that P is a unitary matrix, so we can denote

$$P^{T}LP = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \hline & * & & \\ & \vdots & & \\ & * & & & \\ & & * & & \\ \end{bmatrix}$$

Since $\mathcal{G}(\mathcal{A})$ has a spanning tree, by Lemma 1, L has only one 0 eigenvalue and other eigenvalues have positive real part. This implies $-L_1$ is Hurwitz stable. Now define $\tilde{e}(t) = (P^T \otimes I_m)e(t)$. From (13), we have

$$\dot{\tilde{e}}(t) = -(P^T L P \otimes I_m) \tilde{e}(t) + \left(\left(P^T - \frac{P^T \mathbf{1}_n w^T}{\mathbf{1}_n^T w} \right) \otimes I_m \right) \phi(t).$$
(14)

Let $\tilde{e}(t) = [\tilde{e}_1^T(t), \tilde{e}_2^T(t)]^T$, where $\tilde{e}_1(t) \in \mathbb{R}^m$ and $\tilde{e}_2(t) \in \mathbb{R}^{(n-1)m}$. By (14), we have

$$\dot{\tilde{e}}_1(t) = \left(\left(\frac{1}{\sqrt{w^T w}} w^T - \frac{\frac{1}{\sqrt{w^T w}} w^T \mathbf{1}_n w^T}{\mathbf{1}_n^T w} \right) \otimes I_m \right) \phi(t) = 0.$$

Note that $\tilde{e}_1(t) = \frac{1}{\sqrt{w^T w}} (w^T \otimes I_m) e(t) = \frac{1}{\sqrt{w^T w}} (w^T \otimes I_m) \left(\left(I_n - \frac{\mathbf{1}_n w^T}{\mathbf{1}_n^T w} \right) \otimes I_m \right) x(t) = 0$. Thus, it holds $\tilde{e}_1(t) = 0$ for any $t \ge 0$. Moreover, we have

$$\dot{\tilde{e}}_{2} = -\left(L_{1} \otimes I_{m}\right)\tilde{e}_{2} + \left[\begin{array}{c}\left(p_{2}^{T} - \frac{p_{2}^{T}\mathbf{1}_{n}w^{T}}{\mathbf{1}_{n}^{T}w}\right) \otimes I_{m}\\\vdots\\\left(p_{n}^{T} - \frac{p_{n}^{T}\mathbf{1}_{n}w^{T}}{\mathbf{1}_{n}^{T}w}\right) \otimes I_{m}\end{array}\right]\phi(t).$$

Since $\lim_{t\to\infty} \phi(t) = 0$, by Lemma 8, we have $\lim_{t\to\infty} \tilde{e}_2(t) = 0$. This and the fact that $\lim_{t\to\infty} \tilde{e}_1(t) = 0$ imply $\lim_{t\to\infty} e(t) = 0$. This leads to the validity of this result.

Theorem 3: If the fixed graph $\mathcal{G}(\mathcal{A})$ is directed and strongly connected, then MAS (1) with (11) reaches consensus asymptotically, and the consensus state is located in set \mathbf{X}^* .

Proof: Since the graph is strongly connected, by Lemma 2, there exists a vector $w = [w_1 \cdots w_n]^T > 0$ such that $w^T L = 0$. Consider a positive-definite Lyapunov function candidate $V(t) = \frac{1}{2} \sum_{i=1}^n w_i ||x_i(t) - x_0||^2$, where $x_0 \in \mathbf{X}^*$. By the definition of g_i^+ , it holds $g^+(x_0) = ||x_0||_X = 0$. Based on the property of subgradient, we have $\langle x_i(t) - x_0, \nabla g_i^+(x_i(t)) \rangle \ge g_i^+(x_i(t))$. Taking

the derivative of function V(t) with respect to t yields

$$\dot{V}(t) = \sum_{i=1}^{n} w_i \langle x_i(t) - x_0, \dot{x}_i(t) \rangle
= \sum_{i=1}^{n} w_i \langle x_i(t) - x_0, \sum_{i \in N_i(t)} a_{ij}(t)(x_j(t) - x_i(t)) - \tau [x_i(t) - P_{X_i}(x_i(t))] - \tau \nabla g_i^+(x_i(t)) \rangle
= \sum_{i=1}^{n} \sum_{i \in N_i(t)} w_i a_{ij} \langle x_i(t) - x_0, x_j(t) - x_i(t) \rangle
- \tau \sum_{i=1}^{n} w_i \langle x_i(t) - x_0, x(t) - P_{X_i}(x_i(t)) \rangle
- \tau \sum_{i=1}^{n} w_i \langle x_i(t) - x_0, \nabla g_i^+(x_i(t)) \rangle .$$
(15)

Denote $x(t) = [x_1^T(t), \cdots, x_n^T(t)]^T$, we have

$$\sum_{i=1}^{n} \sum_{i \in N_{i}(t)} w_{i} a_{ij} \langle x_{i}(t) - x_{0}, x_{j}(t) - x_{i}(t) \rangle = -(x(t) - (\mathbf{1}_{n} \otimes I_{m}) x_{0})^{T} (WL \otimes I_{m}) x(t)$$

$$= -x^{T}(t) \left(\frac{WL + L^{T}W}{2} \otimes I_{m} \right) x(t)$$

$$+ x_{0}^{T} (w^{T}L \otimes I_{m}) x(t)$$

$$= x^{T}(t) \left(\frac{W(-L) + (-L)^{T}W}{2} \otimes I_{m} \right) x(t)$$

$$= -\sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{w_{i}a_{ij} + w_{j}a_{ji}}{2} ||x_{j}(t) - x_{i}(t)||^{2}$$

$$\leq 0$$

$$(16)$$

where W = diag(w) is a diagonal matrix formed by w and the last equation results from Lemma 7. By Lemma 4, we know $-\langle x_i(t) - x_0, x_i(t) - P_{X_i}(x_i(t)) \rangle \leq -||x_i(t)||_{X_i}^2 \leq 0$. Based on (15) and (16), we have

$$\dot{V}(t) \le -\tau \sum_{i=1}^{n} w_i \|x_i(t)\|_{X_i}^2 - \tau \sum_{i=1}^{n} w_i g_i^+(x_i(t)).$$
(17)

Note that $g_i^+(x_i(t)) \ge 0$. Thus, $\dot{V}(t) \le 0$. Moreover, V(t) is bounded by zero, it can be concluded that V(t) converges and $V(\infty)$ exists, which implies $||x_i(t) - x_0||$ converges and

 $||x_i(t)||$ is bounded. By (17), we have

$$\tau \int_{0}^{\infty} \sum_{i=1}^{n} w_{i} \|x_{i}(t)\|_{X_{i}}^{2} d_{t} + \tau \int_{0}^{\infty} \sum_{i=1}^{n} w_{i} g_{i}^{+}(x_{i}(t)) d_{t}$$

$$\leq V(0) - V(\infty)$$

$$< \infty.$$

Thus, it holds $\int_0^\infty \|x_i(t)\|_{X_i}^2 d_t < \infty$ and $\int_0^\infty g_i^+(x_i(t))d_t < \infty$. These imply $\lim_{t\to\infty} \|x_i(t) - P_{X_i}(x_i(t))\| = \lim_{t\to\infty} g_i^+(x_i(t)) = 0$ for each $i \in \mathcal{V}$. By the definition of the subgradient $\nabla g_i^+(\cdot)$, we can conclude $\lim_{t\to\infty} \phi_i(t) = 0$ for $i \in \mathcal{V}$. By the continuity of $g_i^+(x_i(t))$ and the boundedness of $\|x_i(t)\|$, it can be concluded $\phi_i(t)$ is bounded. Recall Proposition 1, we know MAS (1) with (11) reaches consensus asymptotically, denote x^* as the consensus state, i.e., $\lim_{t\to\infty} x_i(t) = x^*$ for each $i \in \mathcal{V}$. Therefore, $x^* \in \mathbf{X}^*$. The validity of this result is verified.

Remark 3: The strongly connected condition proposed in Theorem 3 is sufficient to solve CFP (3). In fact, it is also necessary in many cases. Now we set an example to illustrate that the CFP can not be solved by the MAS if the graph is not strongly connected. Suppose graph \mathcal{G} is not strongly connected, then there exists at least one strongly connected component that can not receive information from others. We denote the set consisting of all agents in this component by \mathcal{V}_1 . Suppose that all agents in \mathcal{V}_1 are constrained by inequality $x \leq 0$. If we set $x_i(0) = 0$ for each $i \in \mathcal{V}_1$, then it holds $x_i(t) = 0$ for any t > 0 and $i \in \mathcal{V}_1$. In another strongly connected component that is constrained by inequality $x \leq -1$, it is easy to see that the CFP can never be solved under such a graph.

If communication graph $\mathcal{G}(\mathcal{A})$ is bidirectional and $a_{ij} = a_{ji}$ for each $i \in \mathcal{V}$, $\mathcal{G}(\mathcal{A})$ becomes an undirected graph. For the undirected case, we state the result as follows.

Corollary 2: If the fixed graph $\mathcal{G}(\mathcal{A})$ is undirected and connected, then MAS (1) with (11) reaches consensus asymptotically, and the consensus state is in set \mathbf{X}^* .

B. Convergence under the time-varying communication graph

For system (12), by the properties of linear systems [39], the solution of system (12) can be written as follows.

$$x(t) = \left(\Phi(t,s) \otimes I_m\right) x(s) + \int_s^t \left(\Phi(t,\tau) \otimes I_m\right) u(\tau) d_\tau$$
(18)

where $\Phi(t,s) \otimes I_m$ is the state-transition matrix from state x(s) to state x(t) with $t \ge s \ge 0$. Now, for time-varying graph $\mathcal{G}(t)$, the following assumptions are given.

Assumption 3: The communication graph $\mathcal{G}(t)$ is balanced.

Assumption 4: The δ -digraph $\mathcal{G}_{(\delta,T)}$ is strongly connected.

Lemma 9: [37] Under Assumptions 3 and 4, for any $t \ge s \ge 0$, $\Phi(t, s)$ in (18) satisfies the following inequality

$$\left| \left[\Phi(t,s) \right]_{ij} - \frac{1}{n} \right| \le \gamma^{t-s}, \qquad i, j \in \{1, \cdots, n\}$$

$$\tag{19}$$

where $\gamma = \left(1 - \frac{1}{(8n^2)^{\lfloor n/2 \rfloor}}\right)^{\frac{1}{(\lfloor 1/\delta \rfloor + 1) \lfloor n/2 \rfloor T}}$, the operator $\lfloor x \rfloor$ denotes the largest integer not larger than the value of x.

Proposition 2: Under Assumptions 3 and 4, if $\|\phi_i(t)\| < \infty$ and $\lim_{t\to\infty} \phi_i(t) = 0$ in (11), $i \in \mathcal{V}$, then MAS (1) with (11) reaches consensus asymptotically.

Proof: Since $\mathcal{G}(t)$ is balanced, by Peano-Baker formula (see [39] for detail), it can be concluded that $\Phi(t,s)$ is a double stochastic matrix. Denote $\bar{x}(t) = \frac{1}{n} \sum_{i=1}^{n} x_i(t)$, by (18), we have

$$\bar{x}(t) = \frac{1}{n} \left(\mathbf{1}_n^T \otimes I_m \right) x(s) + \frac{1}{n} \int_s^t \left(\mathbf{1}_n^T \otimes I_m \right) u(\tau) d_\tau.$$
(20)

Based on (18) and (20), we have

$$x(t) - \frac{1}{n} \left(\mathbf{1}_{n} \otimes I_{m} \right) \bar{x}(t) = \left(\left(\Phi(t,0) - \frac{1}{n} \mathbf{1}_{n} \mathbf{1}_{n}^{T} \right) \otimes I_{m} \right) x(0) + \int_{s}^{t} \left(\left(\Phi(t,\tau) - \frac{1}{n} \mathbf{1}_{n} \mathbf{1}_{n}^{T} \right) \otimes I_{m} \right) u(\tau) d_{\tau}.$$

$$(21)$$

Applying (19) in Lemma 9 to equation (21) yields

$$\left\| x(t) - \frac{1}{n} \left(\mathbf{1}_n \otimes I_m \right) \bar{x}(t) \right\| \le \sqrt{mn} \gamma^t \left\| x(0) \right\| + \sqrt{mn} \int_s^t \gamma^{t-\tau} \left\| u(\tau) \right\| d_\tau$$

Since $0 < \gamma = \left(1 - \frac{1}{(8n^2)^{\lfloor n/2 \rfloor}}\right)^{\frac{1}{(\lfloor 1/\delta \rfloor + 1)\lfloor n/2 \rfloor T}} < 1$ and $\lim_{t \to \infty} ||u(t)|| = 0$, by Lemma 6, we have $\lim_{t \to \infty} ||x(t) - \frac{1}{n} (\mathbf{1}_n \otimes I_m) \bar{x}(t)|| = 0$. This leads to the validity of this result.

Theorem 4: Under Assumptions 1, 3 and 4, if $\lim_{t\to\infty} \phi_i(t) = 0$ in (11), $i \in \mathcal{V}$, then MAS (1) with (11) reaches consensus asymptotically, and the consensus state is in set \mathbf{X}^* .

Proof: Consider a positive-definite Lyapunov function candidate $V(t) = \frac{1}{2} \sum_{i=1}^{n} ||x_i(t) - x_0||^2$, where $x_0 \in \mathbf{X}^*$. Taking the derivative of function V(t) with respect to t yields

$$\dot{V}(t) = \sum_{i=1}^{n} \langle x_i(t) - x_0, \dot{x}_i(t) \rangle$$

$$= \sum_{i=1}^{n} \sum_{i \in N_i(t)} a_{ij}(t) \langle x_i(t) - x_0, x_j(t) - x_i(t) \rangle + \sum_{i=1}^{n} \langle x_i(t) - x_0, \phi_i \rangle.$$
(22)

If $\mathcal{G}(t)$ is balanced, we have $\mathbf{1}_n^T L = 0$. This implies that $\sum_{i=1}^n \sum_{i \in N_i(t)} a_{ij}(t) \langle x_i(t) - x_0, x_j(t) - x_i(t) \rangle \leq 0$. The following proof is similar to Theorem 3 and hence it is omitted.

V. DISCRETE-TIME DISTRIBUTED ALGORITHMS FOR SOLVING CFPs

In this section, for discrete-time MAS (2), the following input is presented to solve CFP (3).

$$\begin{cases} u_{i}(t) = h \sum_{j \in N_{i}} a_{ij}(x_{j}(t) - x_{i}(t)) + \phi_{i}(t) \\ \nabla_{i}(t) = \beta(t) \nabla g_{i}^{+}(t) \\ \xi_{i}(t) = x_{i}(t) + h \sum_{j \in N_{i}} a_{ij}(x_{j}(t) - x_{i}(t)) - \nabla_{i}(t) \quad i \in \mathcal{V} \end{cases}$$

$$\varphi_{i}(t) = \alpha(t) \left(\xi_{i}(t) - P_{X_{i}}(\xi_{i}(t))\right) \\ \phi_{i}(t) = -\nabla_{i}(t) - \varphi_{i}(t) \end{cases}$$
(23)

where $\nabla g_i^+(t)$ denotes the subgradient of function $g_i^+(y)$ at $y = x_i(t) + h \sum_{j \in N_i} a_{ij}(x_j(t) - x_i(t))$, h is the control gain to be designed. Note that each agent has only access to the information from its own inequality and set, as well as its own state and the relative states between itself and its neighbors, thus (23) is distributed.

Assumption 5: $\nabla g_i^+(\cdot) \leq K$ for some $K \geq 0, i = 1, \cdots, n$.

Lemma 10: Given a linear system x(t+1) = Ax(t) + u(t), if the state matrix $A \in \mathbb{R}^{n \times n}$ is Schur stable and the control input $u(t) \in \mathbb{R}^n$ is such that $\lim_{t \to \infty} u(t) = 0$, then the linear system is asymptotically stable to zero, i.e., $\lim_{t \to \infty} x(t) = 0$.

Proof: It can be proved by the similar approach in Lemma 8 and using the fact that $\lim_{k\to\infty}\sum_{l=0}^{k}\rho^{k-l}(A) \|u(l)\| = 0 \text{ for } 0 < \rho(A) < 1 \text{, which has been proved in [23].} \qquad \blacksquare$ The properties of graph's Laplacian matrix lead to the following lemmas directly [33].

Lemma 11: For an undirected graph $\mathcal{G}(\mathcal{A})$, if $\mathcal{G}(\mathcal{A})$ is connected and $0 < h < \frac{2}{\lambda_n}$, then it holds $\max_{2 \le i \le n} |1 - h\lambda_i(L)| < 1.$

Lemma 12: For a directed graph $\mathcal{G}(\mathcal{A})$, if $\mathcal{G}(\mathcal{A})$ has a spanning tree and $0 < h < \min_{2 \le i \le n} \frac{2Re(\lambda_i(L))}{|\lambda_i(L)|^2}$,

then it holds $\max_{2 \le i \le n} |1 - h\lambda_i(L)| < 1$. *Proposition 3:* Suppose $\lim_{t \to \infty} \phi_i(t) = 0$ in (23), $i \in \mathcal{V}$, if the undirected graph $\mathcal{G}(\mathcal{A})$ is connected and $0 < h < \frac{2}{\lambda_n}$, then MAS (2) with (23) reaches consensus asymptotically.

Proof: Let $x(t) = [x_1^T(t), \dots, x_n^T(t)]^T$ and $\phi(t) = [\phi_1^T(t), \dots, \phi_n^T(t)]^T$, MAS (2) with (23) can be rewritten as

$$x(t+1) = ((I - hL) \otimes I_m) x(t) + \phi(t).$$
(24)

Denote variable $\bar{x}(t) = \frac{1}{n} \sum_{i=1}^{n} x_i(t) = \frac{1}{n} \left(\mathbf{1}_n^T \otimes I_m \right) x(t)$. Based on (24), we have $\bar{x}(t+1) = \frac{1}{n} \left(\mathbf{1}_n^T \otimes I_m \right) x(t)$. $\bar{x}(t) + u(t)$. Denote $e_i(t) = x_i(t) - \bar{x}(t)$ and $e(t) = [e_1^T(t), \cdots, e_n^T(t)]^T$. Note that if $e(t) \to 0$ as $t \to \infty$, then MAS (2) with (23) reaches consensus asymptotically. From (24), we have

$$e(t+1) = \left((I - hL) \otimes I_m \right) e(t) + \left(\left(I_n - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right) \otimes I_m \right) \phi(t).$$
(25)

Since L is symmetric for \mathcal{G} being undirected. We select $p_i \in \mathbb{R}^n$ such that $p_i^T L = \lambda_i(L)p_i^T$ and form an unitary matrix $P = \begin{bmatrix} \frac{\mathbf{1}_n}{\sqrt{n}}, p_2, \cdots, p_n \end{bmatrix}$ to transform I - hL into a diagonal form $diag(1, (1-h)\lambda_2(L), \cdots, (1-h)\lambda_n(L)) = P^T(I-hL)P$. Denote $\tilde{e}(t) = P^T e(t)$ and partition $\tilde{e}(t)$ into two parts , i.e., $\tilde{e}(t) = [\tilde{e}_1^T(t), \tilde{e}_2^T(t)]^T$. Then, from (25), we have

$$\tilde{e}_1(t+1) = \left(\left(\frac{1}{\sqrt{n}} \mathbf{1}_n^T \left(I_n - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right) \right) \otimes I_m \right) \phi(t)$$

Note that $\left(\frac{1}{\sqrt{n}}\mathbf{1}_n^T \left(I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T\right)\right) \otimes I_m = 0$ and $\tilde{e}_1(t) = \frac{1}{\sqrt{n}} \left(\mathbf{1}_n^T \otimes I_m\right) e(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^n e_i(t) = 0.$ Thus, it holds $\tilde{e}_1(t) = 0$. Moreover, we have

$$\tilde{e}_{2}(t+1) = \Lambda \tilde{e}_{2}(t) + \begin{bmatrix} \left(p_{2}^{T} - \frac{1}{n}p_{2}^{T}\mathbf{1}_{n}\mathbf{1}_{n}^{T}\right) \otimes I_{m} \\ \vdots \\ \left(p_{n}^{T} - \frac{1}{n}p_{n}^{T}\mathbf{1}_{n}\mathbf{1}_{n}^{T}\right) \otimes I_{m} \end{bmatrix} \phi(t)$$

where $\Lambda = diag((1 - h\lambda_2(L))I_m, \cdots, (1 - h\lambda_n(L))I_m)$. By Lemma 11, we know if $0 < h < \frac{2}{\lambda_n}$, Λ is Schur stable. Recalling Lemma 10 yields $\lim_{t\to\infty} \tilde{e}_2(t) = 0$. This and the fact that $\lim_{t\to\infty} \tilde{e}_1(t) = 0$ imply $\lim_{t\to\infty} e(t) = 0$, which leads to the validity of this result.

Proposition 4: Suppose $\lim_{t\to\infty} \phi_i(t) = 0$ in (23), $i \in \mathcal{V}$, if the directed graph $\mathcal{G}(\mathcal{A})$ has a spanning tree and $0 < h < \min_{2\leq i\leq n} \frac{2Re(\lambda_i(L))}{|\lambda_i(L)|^2}$, then MAS (2) with (23) reaches consensus asymptotically.

Proof: It can be proved by replacing the variable $\bar{x}(t)$ in the proof of Proposition 3 with $\hat{x}(t)$ defined in the proof of Proposition 1, and using the fact that $\max_{2 \le i \le n} |1 - h\lambda_i(L)| < 1$ if \mathcal{G} has a spanning tree and $0 < h < \min_{2 \le i \le n} \frac{2Re(\lambda_i(L))}{|\lambda_i(L)|^2}$, which is stated in Lemma 12.

Now we give the convergence condition for (2) with (23) and its proof in detail when the graph is directed.

Theorem 5: Under Assumptions 1 and 5, suppose $\{\alpha(t)\}, \{\beta(t)\}\$ are two sequences such that (a) $\alpha(t) \in [0,1], \sum_{t=0}^{\infty} \alpha(t) \to \infty$ and $\sum_{t=0}^{\infty} \alpha^2(t) < \infty$; (b) $0 \le \beta(t) \le \infty, \sum_{t=0}^{\infty} \beta(t) \to \infty$ and $\sum_{t=0}^{\infty} \beta^2(t) < \infty$.

If the directed graph $\mathcal{G}(\mathcal{A})$ is strongly connected and $0 < h < \rho$, where $\rho = \min\left[\frac{1}{\max_{1 \le i \le n} \left(\sum_{j=1}^{n} a_{ij}\right)},\right]$

 $\min_{1 \le i \le n} \frac{2Re(\lambda_i(L))}{|\lambda_i(L)|^2}$. Then, MAS (2) with (23) reaches consensus asymptotically, and the consensus state is in set **X**^{*}.

Proof: Since the graph is strongly connected, by Lemma 2, there exists a vector $w = [w_1 \cdots w_n]^T > 0$ such that $w^T L = 0$. Submitting (23) to (2), we have

$$x_i(t+1) = \xi_i(t) - \varphi_i(t), \quad i \in \mathcal{V}.$$

Consider the positive-definite Lyapunov function candidate $V(t) = \sum_{i=1}^{n} w_i ||x_i(t) - x_0||^2$, where $x_0 \in \mathbf{X}^*$. Taking the difference of function V(t) yields

$$\begin{aligned} \Delta V(t) &= V(t+1) - V(t) \\ &= \sum_{i=1}^{n} w_i \|\xi_i(t) - \varphi_i(t) - x_0\|^2 - \sum_{i=1}^{n} w_i \|x_i(t) - x_0\|^2 \\ &= \sum_{i=1}^{n} w_i \|(1 - \alpha(t))(\xi_i(t) - x_0) + \alpha(t)(P_{X_i}(\xi_i(t)) - x_0)\|^2 \\ &- \sum_{i=1}^{n} w_i \|x_i(t) - x_0\|^2 \\ &\leq \sum_{i=1}^{n} w_i \Big((1 - \alpha(t)) \|\xi_i(t) - x_0\| + \alpha(t) \|P_{X_i}(\xi_i(t)) - x_0)\| \Big)^2 \end{aligned}$$

$$-\sum_{i=1}^{n} w_{i} \|x_{i}(t) - x_{0}\|^{2}$$

$$\leq \sum_{i=1}^{n} w_{i} \|\xi_{i}(t) - x_{0}\|^{2} - \sum_{i=1}^{n} w_{i} \|x_{i}(t) - x_{0}\|^{2}$$

$$= \sum_{i=1}^{n} w_{i} \|y_{i}(t) - x_{0}\|^{2} - \sum_{i=1}^{n} w_{i} \langle \nabla_{i}(t), y_{i}(t) - x_{0} \rangle$$

$$+ \sum_{i=1}^{n} w_{i} \|\nabla_{i}(t)\|^{2} - \sum_{i=1}^{n} w_{i} \|x_{i}(t) - x_{0}\|^{2}$$
(26)

where $y_i(t) = x_i(t) + h \sum_{j \in N_i} a_{ij}(x_j(t) - x_i(t))$ and the last inequality follows form using the non-expansiveness property of projection operator, i.e., $||P_{X_i}(\xi_i(t)) - x_0|| \le ||\xi_i(t) - x_0||$. Since $\nabla g_i^+(t)$ denotes the subgradient of function $g_i^+(y)$ at $y = y_i(t)$, we have

$$-\langle \nabla_i(t), y_i(t) - x_0 \rangle \le -\beta(t)g_i^+(y_i(t)) \le 0.$$
(27)

Moreover, since $0 < h < \frac{1}{\max_{1 \le i \le n} \left(\sum_{j=1}^{n} a_{ij}\right)}$, we have $0 < 1 - h \sum_{j=1}^{n} a_{ij} < 1$. By the convexity of the norm square function, it holds

$$||y_i(t) - x_0||^2 = ||(1 - h\sum_{j \in N_i} l_{ij})(x_i(t) - x_0) + h\sum_{j \in N_i} l_{ij}(x_j(t) - x_0)||^2$$

$$\leq (1 - h\sum_{j \in N_i} l_{ij})||x_i(t) - x_0||^2 + h\sum_{j \in N_i} l_{ij}||x_j(t) - x_0||^2.$$

Thus, we have

$$\sum_{i=1}^{n} w_{i} \|y_{i}(t) - x_{0}\|^{2} \leq \sum_{i=1}^{n} w_{i}(1 - h\sum_{j \in N_{i}} l_{ij}) \|x_{i}(t) - x_{0}\|^{2} + h\sum_{i=1}^{n} w_{i}\sum_{j \in N_{i}} l_{ij} \|x_{j}(t) - x_{0}\|^{2} = \sum_{i=1}^{n} w_{i} \|x_{i}(t) - x_{0}\|^{2} - h\sum_{i=1}^{n} w_{i}\left(\sum_{j=1}^{n} l_{ij}\right) \|x_{i}(t) - x_{0}\|^{2} + h\sum_{j=1}^{n} \left(\sum_{i=1}^{n} w_{i} l_{ij}\right) \|x_{j}(t) - x_{0}\|^{2} = \sum_{i=1}^{n} w_{i} \|x_{i}(t) - x_{0}\|^{2}$$

$$(28)$$

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where the last equation results from the fact that $\sum_{j=1}^{n} l_{ij} = 0$ and $\sum_{i=1}^{n} w_i l_{ij} = 0$. Submitting (27) and (28) into (26) yields

$$\Delta V(t) \le -\beta(t) \sum_{i=1}^{n} w_i g_i^+(y_i(t)) + \sum_{i=1}^{n} w_i \|\nabla_i(t)\|^2.$$
(29)

From (29), we have $V(t) \leq V(0) + \sum_{t=0}^{t-1} \beta^2(t)(w^T \mathbf{1}_n) K \leq V(0) + \sum_{t=0}^{\infty} \beta^2(t)(w^T \mathbf{1}_n) K < \infty$. By the definition of V(t), it can be concluded that $x_i(t)$ is bounded. By the fact that $\|\nabla_i(t)\| < \infty$, we know $\|\xi_i(t)\| < \infty$, this and the continuity of $P_{X_i}(\xi_i)$ imply $\|\xi_i(t) - P_{X_i}(\xi_i(t))\| < \infty$. Thus, $\lim_{t\to\infty} \varphi_i(t) = 0$. Since graph $\mathcal{G}(\mathcal{A})$ is strongly connected and $0 < h < \min_{2\leq i\leq n} \frac{2Re(\lambda_i(L))}{|\lambda_i(L)|^2}$, from Proposition 4, it can be concluded that MAS (2) with (23) reaches consensus asymptotically, i.e., $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$ for all $i, j \in \mathcal{V}$. Moreover, similar to (26), we have

$$\begin{split} \Delta V(t) &= V(t+1) - V(t) \\ &= \sum_{i=1}^{n} w_i \|y_i(t) - x_0 - \nabla_i(t) - \varphi_i(t)\|^2 - \sum_{i=1}^{n} w_i \|x_i(t) - x_0\|^2 \\ &= \sum_{i=1}^{n} w_i \|y_i(t) - x_0 - \nabla_i(t) - \varphi_i(t)\|^2 - \sum_{i=1}^{n} w_i \|x_i(t) - x_0\|^2 \\ &= \sum_{i=1}^{n} w_i \|y_i(t) - x_0\|^2 - 2\sum_{i=1}^{n} w_i \langle \nabla_i(t) + \varphi_i(t), y_i(t) - x_0 \rangle \\ &+ \sum_{i=1}^{n} w_i \|\nabla_i(t) + \varphi_i(t)\|^2 - \sum_{i=1}^{n} w_i \|x_i(t) - x_0\|^2 \\ &\leq -2\sum_{i=1}^{n} w_i \langle \nabla_i(t) + \varphi_i(t), y_i(t) - x_0 \rangle + \sum_{i=1}^{n} w_i \|\nabla_i(t) + \varphi_i(t)\|^2 \\ &= -2\sum_{i=1}^{n} w_i \langle \nabla_i(t), y_i(t) - x_0 \rangle - 2\sum_{i=1}^{n} w_i \langle \varphi_i(t), \xi_i(t) - x_0 \rangle \\ &- 2\sum_{i=1}^{n} w_i \langle \nabla_i(t), y_i(t) - x_0 \rangle - 2\sum_{i=1}^{n} w_i \langle \varphi_i(t), \xi_i(t) - x_0 \rangle \\ &+ \sum_{i=1}^{n} w_i \left(\|\nabla_i(t)\|^2 + \|\varphi_i(t)\|^2 \right) \end{split}$$

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$$\leq -2\beta(t)\sum_{i=1}^{n} w_{i}g_{i}^{+}(x_{i}(t)) - 2\alpha(t)\sum_{i=1}^{n} w_{i}\|\xi_{i}(t)\|_{X_{i}}^{2} + \sum_{i=1}^{n} w_{i}\left(\|\nabla_{i}(t)\|^{2} + \|\varphi_{i}(t)\|^{2}\right)$$
(30)

where the first inequality results directly from (28). Note that $\sum_{t=0}^{\infty} \sum_{i=1}^{n} w_i \left(\|\nabla_i(t)\|^2 + \|\varphi_i(t)\|^2 \right) < \infty$ and $-2\beta(t) \sum_{i=1}^{n} w_i g_i^+(x_i(t)) - 2\alpha(t) \sum_{i=1}^{n} w_i \|\xi_i(t)\|_{X_i}^2 < 0$. By Lemma 5 and the fact that $\lim_{t \to \infty} x_i(t) = \lim_{t \to \infty} x_j(t), \text{ it can be concluded } V(t) \text{ converges and it holds } \sum_{t=0}^{\infty} (\beta(t) \sum_{i=1}^{n} w_i g_i^+(x_i(t)) + \alpha(t) \sum_{i=1}^{n} w_i \|\xi_i(t)\|_{X_i}^2) < \infty$. Since $\beta(t)g_i^+(x_i(t)) > 0$ and $\alpha(t)\|\xi_i(t)\|_{X_i}^2 > 0$ for all t > 0 and $i = 1, \cdots, n$, we have $\sum_{t=0}^{\infty} \beta(t)g_i^+(x_i(t)) < \infty$ and $\sum_{t=0}^{\infty} \alpha(t)\|\xi_i(t)\|_{X_i}^2 < \infty$. By the facts $\sum_{t=0}^{\infty} \alpha(t) \to \infty$ and $\sum_{t=0}^{\infty} \beta(t) \to \infty$, we have $\lim_{t \to \infty} \inf \|\xi_i(t) - P_{X_i}(\xi_i(t))\| = \lim_{t \to \infty} \inf g_i^+(x_i(t)) = 0$. Thus, there exists a subsequence $\{x_i(t_k)\}$ of $x_i(t)$ such that $\lim_{k \to \infty} x_i(t_k) = x_i^*$, where x_i^* is a vector such that $g_i^+(x_i^*) = h_i(x_i^*) = 0$ for each $i = 1, \cdots, n$. Recall the fact that $\lim_{t \to \infty} x_i(t) = \lim_{t \to \infty} x_j(t)$, we have $x_i^* = x_j^*$ for any $i, j \in \mathcal{V}$. Let $x^* = x_i^*$, we have $\lim_{k \to \infty} x_i(t_k) = x^*$. By the fact $\sum_{i=1}^{n} w_i \|x_i(t) - x_0\|^2$ converges and $\lim_{t \to \infty} \dot{x}_i(t) = 0$, we can conclude $\lim_{k \to \infty} x_i(t_k) = x^*$. By the fact $\sum_{i=1}^{n} w_i \|x_i(t) - x_0\|^2$ converges and $\lim_{t \to \infty} \dot{x}_i(t) = 0$, we can conclude $\lim_{k \to \infty} x_i(t_k) = x^*$. By the fact $\sum_{i=1}^{n} w_i \|x_i(t) - x_0\|^2$ converges and $\lim_{t \to \infty} \dot{x}_i(t) = 0$, we can conclude $\lim_{k \to \infty} x_i(t_k) = x^*$. By the fact $\sum_{i=1}^{n} w_i \|x_i(t) - x_0\|^2$ converges and $\lim_{t \to \infty} \dot{x}_i(t) = 0$ for any $i \in \mathcal{V}$. This means $x \in X = \bigcap_{i=1}^{n} X_i$. Therefore, x^* is a feasible solution to CFP (3), i.e., $x^* \in \mathbf{X}^*$.

Corollary 3: Under Assumptions 1 and 5, suppose $\{\alpha(t)\}$, $\{\beta(t)\}$ are two sequences such that

(a)
$$\alpha(t) \in [0,1]$$
, $\sum_{t=0}^{\infty} \alpha(t) \to \infty$ and $\sum_{t=0}^{\infty} \alpha^2(t) < \infty$;
(b) $0 \le \beta(t) \le \infty$, $\sum_{t=0}^{\infty} \beta(t) \to \infty$ and $\sum_{t=0}^{\infty} \beta^2(t) < \infty$

If the graph $\mathcal{G}(\mathcal{A})$ is undirected and strongly connected, $0 < h < \frac{1}{\max_{1 \le i \le n} \sum_{j=1}^{n} a_{ij}}$. Then, MAS (2) with (23) reaches consensus asymptotically, and the consensus state is in set \mathbf{X}^* .

Proof: By Geršgorin Disc theorem, we can conclude $h < \frac{2}{\lambda_N}$ if $h < \frac{1}{\max_{1 \le i \le n} \sum_{j=1}^{n} a_{ij}}$. Together with Lemma 11, it can be proved by using the similar approach to Theorem 5 and hence the proof is omitted.

VI. A SPECIAL CASE: A DISTRIBUTED GRADIENT-BASED ALGORITHM FOR CFPS INVOLVING LINEAR INEQUALITIES

In this section, we will develop a distributed gradient-based algorithm for the CFP as follows.

$$\begin{cases}
A_i x - b_i \leq 0 \\
x \in X : \stackrel{\Delta}{=} \cap_{i=1}^n X_i
\end{cases} \quad i = 1, \cdots, n \tag{31}$$

where $A_i \in \mathbb{R}^{m_i \times r}$ and $b \in \mathbb{R}^{m_i}$. It assumes CFP (31) has a non-empty feasible solution set \mathbf{X}^* . For a vector $y = [y_1, \dots, y_n]^T$, we define $y^+ = [y_1^+, \dots, y_n^+]^T$ and $y^- = [y_1^-, \dots, y_n^-]^T$, where $y_i^+ = \max(y_i, 0)$ and $y_i^- = \min(y_i, 0)$. We introduce a function $\psi(y) = ||y^+||^2$. Note that $\psi(y) = 0$ if and only if $y \leq 0$. The function $\psi(y)$ is convex and differentiable. See the following lemma for detail.

Lemma 13: For any vector $y \in \mathbb{R}^r$, the function $\psi(y) = ||y^+||^2$ is convex, differentiable and its gradient function at point y is $\nabla_y \psi(y) = 2y^+$.

Proof: . For any vector $z \in \mathbb{R}^r$, we have $\psi(y+z) = ||(y+z)^+||^2 = ||y+z-(y+z)^-||^2 \le ||y+z-(y)^-||^2 = ||y^++z||^2 \le \psi(y) + 2[y^+]^T z + ||z||^2$, where the first inequality follows from the fact that $(y+z)^- = \arg\min_{v\leq 0} ||(y+z)-v||$. Moreover, it holds that $\psi(y+z) = ||(y+z)-(y+z)^-||^2 = ||(y^++[y^-+z-(y+z)^-]||^2 \ge \psi(y) + 2[y^+]^T z + ||y^-+z-(y+z)^-||^2 \ge \psi(y) + 2[y^+]^T z$, where the first inequality follows from the fact that it holds that $[y^+]^T y^- = 0$ and $[y^+]^T (y+z)^- \le 0$. Therefore, it holds that $\lim_{\varepsilon \to 0} \frac{\psi(y+\varepsilon\Delta y)-\psi y}{\varepsilon} = 2[y^+]^T \Delta y$. This means $\nabla_y \psi(y) = 2y^+$. From the fact that $\psi(y+z) \ge \psi(y) + 2[y^+]^T z$, we know $\psi(y)$ is convex. ■ Now we present the following distributed gradient-based algorithm for CFP (31).

$$\dot{x}_{i}(t) = \sum_{i \in N_{i}} a_{ij}(x_{j}(t) - x_{i}(t)) - \tau \left(A_{i}^{T} (A_{i}x_{i}(t) - b_{i})^{+} + x_{i}(t) - P_{X_{i}}(x_{i}(t)) \right), \quad i = 1, \cdots, n$$
(32)

where $\tau > 0$ is a positive coefficient, $x_i(t) \in \mathbb{R}^r$ represents the estimation value of the solutions to CFP (31).

Theorem 6: If the graph $\mathcal{G}(\mathcal{A})$ is strongly connected, then $x_i(t)$ in (32) converges to a fixed vector x^* asymptotically for $i = 1, \dots, n$ and x^* is in feasible solution set \mathbf{X}^* of (31).

Proof: By Lemma 13, it is not difficult to prove that the term $A_i^T (A_i x_i - b_i)^+$ is the gradient of function $||(A_i x_i - b_i)^+||^2$. It can also be viewed as the unique subgradient of $||(A_i x_i - b_i)^+||^2$. Then this result can be proved by the same method as Theorem 3 and hence it is omitted.

VII. SIMULATIONS

In this section, we give numerical examples to illustrate the obtained results. Consider a multi-agent system consisting of five agents, the goal of the agents is to cooperatively search a feasibility $z^* = [z_1^*, z_2^*]^T$ of the CPF which includes two closed convex sets $X_1 = \{(z_1, z_2) | 2 \le z_1 \le 4, 0 \le z_2 \le 2\}$ and $X_2 = \{(z_1, z_2) | 2.5 \le z_1 \le 4.5, 1 \le z_2 \le 3\}$, and three linear inequalities $c(z) = 2z_1 - 3z_2 - 2 \le 0$, $d(z) = 2z_1 + 3z_2 - 11 \le 0$ and $q(z) = 8z_1 - 3z_2 - 28 \le 0$. In Fig.1, the yellow region represents the feasible region. Set X_i is only known to agent *i* for i = 1, 2, and agents 3, 4 and 5 can only have access to c(z), d(z), q(z), respectively. In the following, we will present simulation results in three cases: The first two cases are for continuous-time distributed algorithms under the fixed and time-varying graphs, respectively. The third case is for the discrete-time distributed algorithm under the fixed graph. For each case, the communication graph is directed.

We first show the simulation result in the first case. The communication graph is shown in Fig. 2, which is strongly connected. The weight of each edge connecting different agents is 1. Set coefficient $\tau = 20$ and let the initial state of each agent be $x_1(0) = [0, 5]^T, x_2(0) = [3, -2]^T, x_3(0) = [2, 3]^T, x_4(0) = [5, 1]^T, x_5(0) = [2, -3]^T$. The trajectory of MAS (1) with (11) is shown in Fig. 3. All agents also reach consensus at $z^* = [2.58, 1.23]^T$ which is a solution to the CFP. This is consistent with the result established in Theorem 3.

Now, we show the simulation result in the second case, the communication topologies switch between two bidirectional subgraphs depicted in Fig. 4 and the switching law is given by Fig. 5. It is obvious that the δ -graph associated with the time-varying graph is strongly connected. The weight of each edge connecting different agents is also being 1. Set coefficient $\tau = 35$. Under the same initial condition as the first case, the trajectory of MAS (1) with (11) is shown in Fig. 6. All agents reach consensus at $z^* = [2.61, 1.37]^T$ while remaining in the feasible region of the CFP. This is consistent with the result established in Theorem 4.

In addition, we show the simulation result in the third case. The communication topology in the first case is used to conduct this simulation. Set $\alpha(t) = \beta(t) = \frac{1}{0.02t+1}$ and h = 0.25. Under the same initial condition as the last two cases, the trajectory of MAS (2) with (23) is shown in Fig. 7. All agents reach consensus at $z^* = [2.57, 1.54]^T$ which is a solution to the CFP. This accords with the result established in Theorem 5.

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Fig. 1. The feasible region of the CFP.

Fig. 2. The communication graph in the first case.



Fig. 3. The trajectory of the multi-agent system in the first case. Symbol "*" represents the initial states of agents while "o" represents the final states of them.

VIII. CONCLUSIONS

In this paper, the CFPs have been studied for multi-agent systems through local interactions. The distributed control algorithms were designed for both continuous- and discrete-time systems, respectively. In each case, a centralized approach was first introduced to solve the CFP. Then distributed control algorithms were proposed based on the subgradient and projection operations. The conditions associated with connectivity of the communication graph were given to ensure convergence of the distributed algorithms. The results showed that for the continuous-time case, if the communication graph is fixed and strongly connected, the MAS can reach consensus



Fig. 4. The communication graph in the second case, which consists of two subgraphs. The left one is labeled 1 and the right one is labeled 2.



Fig. 5. The switching law of the time-varying graph.



Fig. 6. The trajectory of the multi-agent system in the second case. Symbol "*" represents the initial states of agents while "o" represents the final states of them.

asymptotically and the consensus state is located in the solution set of the CFP. Moreover, the same result can be achieved if the δ -graph associated with a time-varying graph is strongly connected. For the discrete-time case, under the condition of strong connectivity associated with the directed graph, if the control gain h and the step-sizes $\alpha(t)$ and $\beta(t)$ are properly chosen, convergence of the distributed algorithm can also be guaranteed. Furthermore, a distributed gradient-based algorithm has been designed for a special case in which the CFP involves linear inequalities. Finally, simulation examples have been conducted to demonstrate the effectiveness



Fig. 7. The trajectory of the multi-agent system in the third case. Symbol "*" represents the initial states of agents while "o" represents the final states of them.

of our results. Our future work will focus on some other interesting topics, such as the case under quantization, time delays, packet loss and communication bandwidth constraints, which will bring new challenges in solving CFPs over a network of agents.

REFERENCES

- Y. Guan, Z. Ji, L. Zhang, L.Wang. Decentralized stabilizability of multi-agent systems under fixed and switching topologies. Systems & Control Letters, vol. 62, no. 5, pp. 438-446, 2013.
- [2] J. Ma, Y. Zheng, B. Wu, L. Wang. Equilibrium topology of multi-agent systems with two leaders: A zero-sum game perspective. Automatica, vol. 73, pp. 200-206, 2016.
- [3] J. Ma, Y. Zheng, L. Wang. Topology selection for multi-agent systems with opposite leaders. Systems & Control Letters, vol. 93, pp. 43-49, 2016.
- [4] G. Jing, Y. Zheng, L. Wang. Consensus of multiagent systems with distance-dependent communication networks. IEEE Transactions on Neural Networks and Learning Systems, DOI: 10.1109/TNNLS.2016.2598355, 2016.
- [5] W. Ren, R. Beard. Consensus seeking in multi-agent systems under dynamically changing interaction topologies. IEEE Transactions on Automatic Control, vol. 50, no. 3, pp. 655-661, 2005.
- [6] R. Olfati-Saber, R. Murray. Consensus problems in the networks of agents with switching topology and time delays. IEEE Transactions on Automatic Control, vol. 49, no. 9, pp. 1520-1533, 2004.
- [7] L. Wang, F. Xiao. Finite-time consensus problems for networks of dynamic agents. IEEE Transactions on Automatic Control, vol. 55, no. 4, pp. 950-955, 2010.
- [8] Y. Zheng, L. Wang. Finite-time consensus of heterogeneous multi-agent systems with and without velocity measurements. Systems & Control Letters, vol. 61, no. 8, pp. 871-878, 2012.
- [9] F. Xiao and L. Wang. Asynchronous consensus in continuous-time multi-agent systems with switching topology and timevarying delays. IEEE Transactions on Automatic Control, vol. 53, no. 8, pp. 1804-1816, 2008.

- [10] Z. Zhang, L. Zhang, F. Hao, L. Wang. Leader-Following Consensus for Linear and Lipschitz Nonlinear Multiagent Systems With Quantized Communication. IEEE Transactions on Cybernetics, DOI: 10.1109/TCYB.2016.2580163, 2016.
- [11] Y. Tang, H. Gao, W. Zou, J. Kurths. Distributed synchronization in networks of agent systems with nonlinearities and random switchings. IEEE Transactions on Cybernetics, vol. 43, no. 1, pp. 358-370, 2013.
- [12] S. Kar, J. M. F. Moura, K. Ramanan. Distributed parameter estimation in sensor networks: nonlinear observation models and imperfect communication. IEEE Transactions on Information Theory, vol. 58, no. 6, pp. 3575-3605, 2012.
- [13] P. Lin, W. Ren, Y. Song. Distributed multi-agent optimization subject to nonidentical constraints and communication delays. Automatica, vol. 65, pp. 120-131, 2016.
- [14] A. Nedić, A. Olshevsky. Distributed optimization over time-varying directed graphs. IEEE Transactions on Automatic Control, vol.60, no.3, pp. 601-615, 2015.
- [15] S. Rahili, W. Ren. Distributed continuous-time convex optimization with time-varying cost functions. IEEE Transactions on Automatic Control, DOI 10.1109/TAC.2016.2593899, 2016.
- [16] S. Mou, J. Liu, A. S. Morse. A distributed algorithm for solving a linear algebraic equation. IEEE Transactions on Automatic Control, vol. 60, no. 11, pp. 2863-2878, 2015.
- [17] H. Cao, T. E. Gibson, S. Mou, Y. Liu. Impacts of network topology on the performance of a distributed algorithm solving linear equations. arXiv preprint arXiv: 1603.04154, 2016.
- [18] S. Martin, J. M. Hendrickx. Continuous-time consensus under non-instantaneous reciprocity. IEEE Transactions on Automatic Control, vol. 61, no. 9, pp. 2484-2495, 2016.
- [19] J. M. Hendrickx, J. N. Tsitsiklis. Convergence of type-symmetric and cut-balanced consensus seeking systems. IEEE Transactions on Automatic Control, vol. 58, no. 1, pp. 214-218, 2013.
- [20] A. Jadbabaie, J. Lin, A. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. IEEE Transactions on Automatic Control, vol. 48, no. 6, pp. 988-1001, 2003.
- [21] L. Moreau. Stability of multiagent systems with time-dependent communication links. IEEE Transactions on Automatic Control, vol. 50, no. 2, pp. 169-182, 2005.
- [22] A. Nedić, A. Ozdaglar, P. A. Parrilo. Constrained consensus and optimization in multi-agent networks. IEEE Transactions on Automatic Control, vol. 55, no. 4, pp. 922-938, 2010.
- [23] S. S. Ram, A. Nedic, V. V. Veeravalli. Distributed stochastic subgradient projection algorithms for convex optimization. Journal of Optimization Theory and Applications, vol.147, pp. 516-545, 2010.
- [24] P. Lin, W. Ren. Constrained consensus in unbalanced networks with communication delays. IEEE Transactions on Automatic Control, vol. 59, no. 3, PP. 775-781, 2014.
- [25] G. Shi, K. H. Johansson, Y. Hong. Reaching an optimal consensus: dynamical systems that compute intersections of convex sets. IEEE Transactions on Automatic Control, vol. 58, no.3, pp. 610-622, 2013.
- [26] G. Shi, B. D. O. Anderson. Distributed network flows solving linear algebraic equations. 2016 American Control Conference (ACC). IEEE, pp. 2864-2869, 2016.
- [27] T. M. Cover. Geometrical and statistical properties of systems of linear inequalities with applications in pattern recognition. IEEE transactions on electronic computers, vol. EC-14, pp. 326-334, 1965.
- [28] I. Yamada, K. Slavakis, K.Yamada. An efficient robust adaptive filtering algorithm based on parallel subgradient projection techniques. IEEE Transactions on Signal Processing, vol. 50, no. 5, pp. 1091-1101, 2002.
- [29] Y. Censor, M. D. Altschuler, W. D. Powlis. On the use of Cimminos simultaneous projections method for computing a solution of the inverse problem in radiation therapy treatment planning. Inverse Problems, vol. 4, pp. 607-623, 1998.

- [30] G. T. Herman. Image reconstruction from projections. Academic Press, New York, 1980.
- [31] S. Han, O. Mangasarian. Exact penalty functions in nonlinear programming. Mathematical programming, vol. 17, no. 1, pp. 251-269, 1979.
- [32] D. Richert, J. Cortés. Robust Distributed Linear Programming. IEEE Transactions on Automatic Control, vol. 60, no. 10, PP. 2567-2582, 2014.
- [33] C. Godsil, G. Royle. Algebraic Graph Theory. New York: Springer, 2001.
- [34] J. Aubin, A. Cellina. Differential Inclusions. Berlin, Germany: Speringer-Verlag, 1984.
- [35] H. Robbins, D. Siegmund. A convergence theorem for nonnegative almost supermartingales and some applications. Optimizing Methods in Statistics, Academic Press, New York, pp. 233-257, 1971.
- [36] B. Touri, B. Gharesifard. Continuous-time distributed convex optimization on time-varying directed networks. 2015 54th IEEE Conference on Decision and Control (CDC), IEEE, pp. 724-729, 2015.
- [37] K. Lu, G. Jing, and L. Wang. Distributed algorithms for solving convex inequalities. arXiv preprint arXiv:1609.03161, 2016.
- [38] B. Touri. Product of random stochastic matrices and distributed averaging. Springer, 2012.
- [39] R. W. Brockett. Finite-Dimensional Linear Systems. New York: Wiley, 1970.