Distributed Algorithms for Searching Generalized Nash Equilibrium of Noncooperative Games

Kaihong Lu, Gangshan Jing, and Long Wang

***Abstract*—In this paper, the distributed Nash equilibrium (NE) searching problem is investigated, where the feasible action sets are constrained by nonlinear inequalities and linear equations. Different from most of the existing investigations on distributed NE searching problems, we consider the case where both cost functions and feasible action sets depend on actions of all players, and each player can only have access to the information of its neighbors. To address this problem, a continuous-time distributed gradient-based projected algorithm is proposed, where a leader-following consensus algorithm is employed for each player to estimate actions of others. Under mild assumptions on cost functions and graphs, it is shown that players' actions asymptotically converge to a generalized NE. Simulation examples are presented to demonstrate the effectiveness of the theoretical results.**

***Index Terms*—Consensus, distributed algorithm, Nash equilibrium (NE), noncooperative game.**

I. Introduction

In NONCOOPERATIVE games, Nash equilibrium (NE) is one of the most important solution concepts. If the feasible action set of each player depends on the actions of the other players, the solution to a noncooperative game is referred to as generalized NE (GNE) [1]. The problems of searching NEs or GNEs have received increasing attention in recent years. This is due to its widely practical applications including power systems [2], social science [3], and radio networks [4].

Vast results on searching NEs have been achieved. For example, a gradient method was employed for finding differential NEs of continuous games in [5]. For potential games, learning algorithms were studied to search NEs in [9] and [10], and an extremum seeking-based algorithm was presented in [11]. Moreover, strategies were designed for solving GNE problems in [12]–[14]. Unfortunately, all of the aforementioned works are conducted by centralized approaches.

Recently, along with the penetration of multiagent networks [15]–[22], distributed algorithms for seeking NEs in noncooperative games have been investigated. In [24], distributed iterative regularization algorithms were investigated for monotone games. In [25], a distributed extremum seeking-based algorithm was proposed to search NEs. In [26], based on saddle point strategy, a continuous-time distributed algorithm was proposed for seeking an NE in a two-network zero-sum game. In [27], with coupled linear constraints considered, a discrete-time distributed primal-dual algorithm was proposed to seek GNEs, where players make decisions by exchanging multipliers with their neighbors through a network. In [28], distributed stochastic gradient strategies were developed for seeking a random NE. In [29], with transmission delays considered, distributed gradient-based computation algorithms were presented for seeking GNEs. Works in [24]–[29] assume that players' cost functions are only determined by their neighbors' actions, or require real actions of their opponents. However, costs or payoffs of players are usually affected not only by their neighbors' actions, but also by the other players', and achieving full information is often impractical in many engineering systems, such as distributed sensor networks [18] and mobile ad-hoc networks [33].

These days, considering the case where players' cost functions depend on all players' actions, the problems of distributively searching NEs through a multiagent network have been investigated [30]–[34]. Distributed strategies were designed to find an NE of aggregative games in [30] and [31], where each player's cost function depends on its own action and an aggregate of all players' actions. With coupled linear equation constraints considered, a continuous-time distributed strategy based on projected dynamics and tracking dynamics is designed for searching a GNE in [32], where a nonsmooth average consensus-tracking algorithm is used to compute the aggregate of the game. However, for a general class of games where players' cost functions are not coupled through a common term, the aggregate of actions is not enough for players to make decisions. An asynchronous gossip-based algorithm was proposed for searching an NE without using full information from all players in [33], and continuous-time distributed NE-seeking strategies were investigated in [34].

Investigations [33], [34] involve the general class of games without coupled constraints. In applications when players compete for shared network resources [2]–[4], players usually have not only private constraints determined by their own actions, but also coupled or shared constraints that are determined by their opponents' actions. Thus, it is significant to study general games with both private and coupled constraints.

Manuscript received October 4, 2017; revised February 7, 2018 and April 9, 2018; accepted April 15, 2018. Date of publication May 2, 2018; date of current version March 28, 2019. This work was supported by the National Natural Science Foundation of China under Grant 61751301 and Grant 61533001. This paper was recommended by Associate Editor S. E. Shimony. *(Corresponding author: Long Wang.)*

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Digital Object Identifier 10.1109/TCYB.2018.2828118

Motivated by the observations above, in this paper, we first consider the problem of distributively searching a GNE of noncooperative games via a multiagent network, where both private and coupled constraints are involved. Then, we propose a novel continuous-time distributed gradient-based projected (DGP) algorithm to address this problem. Finally, the convergence of the continuous-time DGP algorithm is proved. The novelty and contributions are summarized as follows.

1) Each player's private constraints are modeled by convex inequalities, and the coupled constraint is modeled by a linear equation that is shared by all players. Compared with works [30]–[32], the game considered by us is general, in which the couplings in the cost functions need not to be a common aggregate. Different from [33] and [34], here players' feasible action sets are coupled across actions of their opponents. Owing to the coexistence of nonlinear inequality and coupled equation constraints, challenges emerge in developing algorithms to solve the suboptimization problems that players aim to selfishly minimize their own cost functions.
2) In the presented algorithm, a leader-following consensus algorithm is employed for each player to estimate actions of others, and a consensus-based estimator is employed for each player to estimate the optimal multiplier associated with the coupled equation. Different from [32], where the algorithm involves a sufficiently large and fixed control gain, our algorithm relies on a diminishing gain, which helps ensure the convergence of the algorithm and reduce the computing cost. By implementing the proposed algorithm, each player makes decisions by using the information associated with its own cost function, its own private constraint and the coupled constraint, its own action, and actions and estimates received from its neighbors.
3) The convergence of the continuous-time DGP algorithm is proved by using convex analysis theory, consensus theory, and Lyapunov stability theory. The result shows that if the graph is undirected and connected, the proposed algorithm enables the players' actions to asymptotically converge to a GNE. Different from the continuous-time distributed algorithm for searching GNEs of aggregate games in [32], where the cost functions with a common aggregate term are assumed to be twice differentiable, our results hold even if the cost functions in noncooperative games are not twice differentiable.

The rest of this paper is organized as follows. In Section II, we formulate the problem to be studied and present the continuous-time distributed algorithm for searching a GNE. In Section III, we state our main results and give theoretical proofs in detail. In Section IV, simulation examples are presented. Section V concludes the whole paper.

II. Preliminaries

A. Notations

Throughout this paper, we use $|x|$ to represent the absolute value of scalar x. $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of real numbers and set of non-negative real numbers, respectively. Let $\mathbb{R}^m$ be the m-dimensional real vector space. For a given vector $x \in \mathbb{R}^m$, $\|x\|$ denotes the standard Euclidean norm of x, that is, $\|x\| = \sqrt{x^T x}$. $\mathbf{1}_m$ denotes the m-dimensional vector with elements being all ones. For $a \in \mathbb{R}$, we denote $a^+ = \max(a, 0)$ and $a^- = \min(a, 0)$. For vector $x \in \mathbb{R}^m$, we denote $x^+ = [x_1^+, \dots, x_m^+]^T$ and $x^- = [x_1^-, \dots, x_m^-]^T$, respectively. For differentiable function $f(\cdot) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, we denote the gradient of $f(x)$ with respect to x by $\nabla_x f(x)$. Given a set of vectors $x_i \in \mathbb{R}^{m_i}$, $i = 1, \dots, n$, $\mathrm{diag}(x_1, \dots, x_n)$ denotes a matrix where the ith diagonal block is x_i and other elements are zero. For matrices A and B, the Kronecker product is denoted by $A \otimes B$.

Set $\mathcal{K} \subset \mathbb{R}^m$ is called a convex set if $\gamma x + (1-\gamma) y \in \mathcal{K}$ for any scalar $0 < \gamma < 1$ and $x, y \in \mathcal{K}$. For a closed convex set $\mathcal{K}$, there is a unique element $P_{\mathcal{K}}(x) \in \mathcal{K}$ such that $\|x - P_{\mathcal{K}}(x)\| = \inf_{y \in \mathcal{K}} \|x - y\|$, where $P_{\mathcal{K}}(\cdot)$ is called the projection onto the set $\mathcal{K}$.

Lemma 1 [39]: If $\mathcal{K} \subset \mathbb{R}^m$ is a closed convex set, then for any $u \in \mathbb{R}^m$ and $v \in \mathcal{K}$, the following inequality holds:

$$(u - P_{\mathcal{K}}(u))^T (P_{\mathcal{K}}(u) - v) \geq 0. \tag{1}$$

B. Problem Formulation

Consider a game $\Gamma(\mathcal{V}, \Omega, J)$ with n players in a communication graph $\mathcal{G}(\mathcal{A})$. $\mathcal{V} = \{1, \dots, n\}$ represents the set of players; $\Omega = \Omega_1 \times \cdots \times \Omega_n$ denotes the action set of players, where $\Omega_i \subset \mathbb{R}^m$ is the action set of player i; $J = (J_1, \dots, J_n)$, where J_i is the cost function of player i; and $\mathcal{A}$ is the weighted adjacency matrix of $\mathcal{G}(\mathcal{A})$. Let $x = (x_i, x_{-i}) \subset \Omega$ denote all players' actions, where x_i is the action of player i and x_{-i} denotes actions of players other than player i, that is, $x_{-i} = [x_1^T, \dots, x_{i-1}^T, x_{i+1}^T, \dots, x_n^T]^T$. For game $\Gamma(\mathcal{V}, \Omega, J)$, an action profile $x^* = (x_i^*, x_{-i}^*)$ is called the NE of this game if and only if $J_i(x_i^*, x_{-i}^*) \leq J_i(x_i, x_{-i}^*)$ holds for any $x_i \in \Omega_i$ and $i \in \mathcal{V}$.

In this paper, we consider game $\Gamma(\mathcal{V}, \Omega, J)$ with both private and coupled constraints, and the feasible action set of all players is defined as $\chi := \chi^c \cap (\chi_1^p \times \cdots \times \chi_n^p)$, where χ^c represents the coupled constraint defined as $\chi^c = \{x \in \Omega | h(x) = 0\}$, where $h(x) = c^T x - d$, $c = [c_1^T, \dots, c_n^T]^T$, $c_i \in \mathbb{R}^m$ and $d \in \mathbb{R}$; χ_i^p denotes player i's private constraint defined as $\chi_i^p = \{x_i \in \Omega_i | g_i(x_i) \leq 0\}$ and function $g_i(\cdot) : \mathbb{R}^m \to \mathbb{R}$ can be nonlinear for any $i \in \mathcal{V}$. Accordingly, player i's feasible action set is denoted by $\chi_i(x_{-i}) = \{x_i | (x_i, x_{-i}) \in \chi\}$. To find an optimal action in its feasible action set, minimize its own cost function subject to a constraint

$$\begin{aligned} \min_{x_i} \quad & J_i(x_i, x_{-i}) \\ \text{subject to} \quad & x_i \in \chi_i(x_{-i}). \end{aligned} \tag{2}$$

Since each $\chi_i(x_{-i})$ depends on other players' actions, (2) can be viewed as a GNE problem, and we refer the reader to [1] for detail. Note that game $\Gamma(\mathcal{V}, \Omega, J)$ considered by us is general, where the cost functions in (2) need not to be in the form of aggregative games [30]–[32]. Some basic assumptions on the cost functions, which are also made in [33], are given as follows.

Assumption 1: For $i \in \mathcal{V}$, $\Omega_i \in \mathbb{R}^m$ is a nonempty, compact and convex set; $J_i(x_i, x_{-i})$ is continuous for $x \in \Omega$, differentiable, and convex with respect to x_i for any $x_{-i} \in \mathbb{R}^{(n-1)m}$; $g_i(y)$ is differentiable and convex for any $y \in \mathbb{R}^m$; and the feasible action set χ is nonempty.

In Assumption 1, the compactness of Ω_i means that there exists some constant $K_i > 0$ such that $\|\xi_i\| \leq K_i$ for any $\xi_i \in \Omega_i$ [45]. Different from [32], here each $J_i(x_i, x_{-i})$ is unnecessary to be twice differentiable with respect to x_i. Define $\Psi(x) = [\nabla_{x_1} J_1{}^T(x), \ldots, \nabla_{x_n} J_n{}^T(x)]^T$, which is usually called pseudo-gradient mapping [33].

Assumption 2: Assumptions on strong monotonicity and Lipschitz continuity of cost functions are made as follows.

1) *Strong Monotonicity:* $(\Psi(x)-\Psi(y))^T(x-y) \geq \mu\|x-y\|^2$ for some $\mu > 0$, $\forall\ x,\ y \in \Omega$.
2) *Lipschitz Continuity:* $\|\nabla_{x_i} J_i(x_i, w) - \nabla_{x_i} J_i(x_i, z)\| \leq \gamma\|w-z\|$ for some constant $0 < \gamma < (2\sqrt{\mu}/\sqrt{n})$, $\forall\ x_i \in \Omega_i;\ w, z \in \mathbb{R}^{(n-1)m}$, $i \in \mathcal{V}$.

To ensure that the information on each player's action can reach all others, the following connectivity assumption associated with communication graph $\mathcal{G}(\mathcal{A})$ is made.

Assumption 3: $\mathcal{G}(\mathcal{A})$ is undirected and connected.

Problem 1: For game $\Gamma(\mathcal{V}, \Omega, J)$, suppose that player i can only communicate with its neighbors via communication graph $\mathcal{G}(\mathcal{A})$, and has access to the information associated with J_i, Ω_i, g_i, h for any $i \in \mathcal{V}$. The goal of this paper is to design a distributed strategy for the players such that their actions converge to a GNE that is located in the feasible action set χ.

Lemma 2: Under Assumptions 1 and 2, for (2), $x^* = (x_i^*, x_{-i}^*)$ is a GNE if there exist Lagrangian multipliers $y^* \in \mathbb{R}$ and $W^* \in \mathbb{R}^n$ such that

$$\begin{cases} x^* = P_\Omega\big[x^* - (\Psi(x^*) - cy^* + \nabla G(x^*)W^*)\big] \\ c^T x^* - d = 0 \\ W^* = \big[W^* + g(x^*)\big]^+ \end{cases} \tag{3}$$

where $\nabla G(x^*) = \mathrm{diag}(\nabla_{x_1} g_1(x_1^*), \ldots, \nabla_{x_n} g_n(x_n^*))$ and $g(x^*) = [g_1(x_1^*), \ldots, g_n(x_n^*)]^T$.

Proof: Consider the following variational inequality problem:

$$\begin{aligned} &\text{Find} \quad x^* \\ &\text{subject to} \quad \Psi\big(x^*\big)^T\big(x - x^*\big) \geq 0 \quad \text{for all } x \in \chi. \end{aligned} \tag{4}$$

Since $\Psi(x)$ is strongly monotone, by [40, Th. 2.3.3], we know (4) has a unique solution. Based on the Lagrangian duality for variational inequality, we know x^* is a solution of (4) if and only if there exist Lagrangian multipliers $y^*, \omega_i^* \in \mathbb{R}$ such that the following Karush–Kuhn–Tucker (KKT) conditions hold:

$$\begin{cases} \big(z_i - x_i^*\big)^T\big(\nabla_{x_i} J_i(x^*) - y^* c_i \\ \quad + \omega_i^* \nabla_{x_i} g_i\big(x_i^*\big)\big) \geq 0,\ \forall z_i \in \Omega_i \\ c^T x^* - d = 0 \\ \omega_i^* \geq 0,\ g_i\big(x_i^*\big) \leq 0,\ \omega_i^* g_i\big(x_i^*\big) = 0 \end{cases} \tag{5}$$

for any $i \in \mathcal{V}$ and $\tau > 0$. Since χ is convex, by [41, Th. 3.9], we know any solution of (4) is a GNE of (2). Consequently, the solution to (5) is a GNE of (2). From the projection inequality (1) in Lemma 1, we know the first inequality in (5) holds if and only if $x_i^* = P_{\Omega_i}[x_i^* - (\nabla_{x_i} J_i(x^*) - c_i y^* + \omega_i^* \nabla_{x_i} g_i(x_i^*))]$, and the last three conditions in (5) hold if and only if $\omega_i^* = [\omega_i^* + g_i(x_i^*)]^+$. This leads to the validity of the result. ■

For simplicity, we denote the optimal Lagrangian multiplier sets consisting of y^* and W^* that satisfy KKT condition (3) by $\mathbb{Y}^*$ and $\mathbb{W}^*$, respectively.

C. Distributed Algorithms for Searching GNE

For Problem 1, let vector $\mathbf{x}_{-i} = [x_{i1}^T, \ldots, x_{i(i-1)}^T, x_{i(i+1)}^T, \cdots, x_{in}^T]^T$ denote player i's estimates on all players' actions but its own's, where x_{ij} is the player i's estimate on player j's action. For ease, we denote $\mathbf{x}_i = [x_{i1}^T, \ldots, x_{i(i-1)}^T, x_i^T, x_{i(i+1)}^T, \cdots, x_{in}^T]^T$, where x_i is player i's real action. We propose the following DGP algorithm for player i to seek a GNE of game $\Gamma(\mathcal{V}, \Omega, J)$:

$$\begin{cases} \dot{x}_i(t) = -\alpha_t x_i(t) + \alpha_t P_{\Omega_i}\Big[x_i(t) - \Big(\nabla_{x_i} J_i(\mathbf{x}_i(t)) - c_i(y_i(t) \\ \qquad - c^T \mathbf{x}_i(t) + d\Big) + \big[\omega_i(t) + g_i(x_i(t))\big]^+ \nabla_{x_i} g_i(x_i(t))\Big)\Big] \\ \dot{x}_{ij}(t) = \sum\limits_{k \in N_i, k \neq j} a_{ik}\big(x_{kj}(t) - x_{ij}(t)\big) \qquad\qquad i \in \mathcal{V} \\ \qquad + a_{ij}\big(x_j(t) - x_{ij}(t)\big), \quad j \neq i \\ \dot{y}_i(t) = \sum\limits_{k \in N_i} a_{ik}(y_k(t) - y_i(t)) - \frac{n\alpha_t}{2}\Big(c_i{}^T x_i(t) - \frac{d}{n}\Big) \\ \dot{\omega}_i(t) = -\frac{\alpha_t}{2}\omega_i(t) + \frac{\alpha_t}{2}\big[\omega_i + g_i(x_i)\big]^+ \end{cases} \tag{6}$$

where $y_i(t) \in \mathbb{R}$ denotes player i's estimate on the Lagrangian multiplier associated with the coupled equation constraint; $\omega_i(t) \in \mathbb{R}$ denotes player i's estimate on the Lagrangian multipliers associated with its own inequality constraint, where each a_{ik} is the adjacency weight of an edge in graph $\mathcal{G}(\mathcal{A})$; N_i is the set of indices k such that (i, k) is an edge; $x_i(0) = x_i^0 \in \Omega_i$; $\omega_i(0) = \omega_i^0 \in \mathbb{R}_+$; and $\alpha_t > 0$ is a continuous and nonincreasing function of t such that

$$\int_0^\infty \alpha_t d_t \to \infty \quad \text{and} \quad \int_0^\infty \alpha_t^2 d_t < \infty. \tag{7}$$

By implementing algorithm (6), players make decisions by only using their own actions, their neighbors' actions, and estimates. Moreover, for the coupled constraint, instead of using a global multiplier y, local multiplier y_i is used to estimate the optimal multiplier y^*. Thus, algorithm (6) is distributed. In (6), for any $j \neq i$, player i updates estimate x_{ij} on player j's action by a leader-following consensus algorithm [47]–[49], where the real action x_j, $j \in \mathcal{V}$ can be viewed as the leader's state. The estimator dynamics for y_i is motivated by the consensus algorithms [6]–[8], [35]–[38]. And the gradient-based projected dynamics in algorithm (6) is inspired by the traditional gradient-based projected methods for convex programming problems [42]. A similar method has been used for aggregative games in [32], where it does not involve the nonlinear inequality constraints.

Remark 1: From (7), it is obvious that α_t is diminishing, that is, $\lim_{t\to\infty} \alpha_t = 0$. The conditions for gain α_t are inspired by the diminishing step size in discrete-time gradient-based strategy [43], which is also used in [44]–[46]. The diminishing property of α_t plays an important role in driving all $\mathbf{x}_i$, $i \in \mathcal{V}$ and y_i, $i \in \mathcal{V}$ to reach consensus, respectively. In the related work on aggregative games [32], a fixed gain to enlarge the consensus term is used for all estimates on the Lagrangian multiplier to reach consensus. There is a restriction that the gain needs to be larger than the product of the number of players and the maximum value of local functions. Nevertheless, estimating this product is complex and requires global information. Moreover, quite large edge-weights or control gains are necessary when the product is great, which may cause high computation and communication costs. On the contrary, using the easily achieved time-varying gain, the former restriction can be removed. In particular, a suitable choice of α_t is $\alpha_t = (a_0/[t+b_0])$ for any $t \geq 0$, where a_0 and b_0 are two positive constants.

Remark 2: With (6), though the time-varying gain α_t is involved, the solutions of $x_i(t)$ and $\omega_i(t)$ are maintained in Ω_i and $\mathbb{R}_+$, respectively, for any $i \in \mathcal{V}$. For any $t \geq 0$, $x_i(t)$ and $\omega_i(t)$ can be computed as follows:

$$
\begin{aligned}
x_i(t) = \exp\Big(-\int_0^t \alpha_s d_s\Big)x_i^0 + \int_0^t \exp\Big(-\int_s^t \alpha_\tau d_\tau\Big)\alpha_s \\
\times P_{\Omega_i}\Big[x_i(s) - \Big(\nabla_{x_i}J_i(\mathbf{x}_i(s)) - c_i\big(y_i(s) - c^T\mathbf{x}_i(s) + d\big) \\
+ \big[\omega_i(s) + g_i(x_i(s))\big]^+\nabla_{x_i}g_i(x_i(s))\Big)\Big]d_s.
\end{aligned}
$$

By the integration mean value theorem and $(d/d_s)(\exp(-\int_s^t \alpha_\tau d_\tau)) = \alpha_s \exp(-\int_s^t \alpha_\tau d_\tau)$, it holds

$$
\begin{aligned}
x_i(t) = \exp\Big(-\int_0^t \alpha_s d_s\Big)x_i^0 + \Big(1 - \exp\Big(-\int_0^t \alpha_s d_s\Big)\Big) \\
\times P_{\Omega_i}\Big[x_i(\theta_i) - \Big(\nabla_{x_i}J_i\big(x_i(\mathbf{x}_i(\theta_i)\big) - c_i\big(y_i(\theta_i) - c^T\mathbf{x}_i(\theta_i) + d\big) \\
+ \big[\omega_i(\theta_i) + g_i(x_i(\theta_i))\big]^+\nabla_{x_i}g_i(x_i(\theta_i))\Big)\Big]
\end{aligned}
$$

for some fixed $\theta_i \in (0, t)$. Since $x_i^0 \in \Omega_i$, by the property of the closed convex set, it holds $x_i(t) \in \Omega_i$ for any $t \geq 0$ and $i \in \mathcal{V}$. Similarly, we can conclude $\omega_i(t) \in \mathbb{R}_+$ for any $i \in \mathcal{V}$.

III. Main Results

In this section, we will analyze the convergence of (6). Let us start this section by stating the main result. For simplicity, we denote the solution to (3) by x^*, which is also a GNE of game for (2).

Theorem 1: Under Assumptions 1–3, by algorithm (6), each $\mathbf{x}_i(t)$, $i \in \mathcal{V}$ asymptotically converges to x^*. Moreover, $y_i(t)$, $i \in \mathcal{V}$, asymptotically converges to a common point y^* and $[\omega_1(t), \ldots, \omega_n(t)]^T$ asymptotically converges to a fixed point W^*, where $y^* \in \mathbb{Y}^*$ and $W^* \in \mathbb{W}^*$.

In the following equation, the time index t is omitted when it is not necessary. Denoting variable $e_{ij} = x_{ij} - x_j$, it follows from (6) that:

$$
\dot{e}_{ij} = \sum_{k\in N_i, k\neq j} a_{ik}\big(e_{kj} - e_{ij}\big) - a_{ij}e_{ij} - \dot{x}_j
$$

for any $i, j \in \mathcal{V}$ and $i \neq j$. Let $e_j = [e_{1j}^T, \ldots, e_{(j-1)j}^T, e_{(j+1)j}^T, \ldots, e_{nj}^T]^T$, then, for any $j \in \mathcal{V}$, we have

$$
\dot{e}_j = -\big(\big(L_j + \Lambda_j\big) \otimes I_m\big)e_j - \dot{x}_j \tag{8}
$$

where $[L_j]_{ik} = -a_{ik}$ if $i \neq k$ and $[L_j]_{ik} = \sum_{k\in N_i, k\neq j} a_{ik}$ if $i = k$, $\Lambda_j = \mathrm{diag}(a_{1j}, \ldots, a_{(j-1)j}, a_{(j+1)j}, \cdots, a_{nj})$. For $j \in \mathcal{V}$, we denote $\underline{\lambda}_j$ as the smallest eigenvalue of matrix $(L_j + \Lambda_j)$. Moreover, for any $t \geq 0$, we denote

$$
Y = \frac{1}{n}\sum_{i=1}^n y_i.
$$

The following lemmas, which are useful to prove the main results, are introduced.

Lemma 3 [46]: Let $b(t)$ be a continuous function, if $\lim_{t\to\infty} b(t) = b$ and $0 < \pi < 1$, then $\lim_{t\to\infty} \int_0^t \pi^{t-s} b(s)ds = -(b/\ln\pi)$.

Lemma 4: Under Assumptions 1 and 3, for any $j \in \mathcal{V}$, the following statements hold:

1) $\lim_{t\to\infty} e_j = 0$;
2) $\lim_{t\to\infty} |y_j - Y| = 0$;
3) $\int_0^\infty \alpha_t \|e_j\| d_t < \infty$.

Proof: 1) From Remark 2, we know $x_j(t) \in \Omega_j$ for any $t \geq 0$. By the compactness of Ω_j in Assumption 1, we have $\|x_j(t)\| \leq K_j$. Together with $P_{\Omega_j}[\,\cdot\,] \in \Omega_j$, it holds that

$$
\begin{aligned}
\big\|\dot{x}_j(t)\big\| &= \alpha_t\big\|-x_j(t) + P_{\Omega_j}\big[\Im_j(t)\big]\big\| \\
&\leq \alpha_t\big(\big\|x_j(t)\big\| + \big\|P_{\Omega_j}\big[\Im_j(t)\big]\big\|\big) \\
&\leq 2\alpha_t K_j
\end{aligned} \tag{9}
$$

where

$$
\begin{aligned}
\Im_j(t) = x_j(t) - \Big(\nabla_{x_j}J_j\big(\mathbf{x}_j(t)\big) - c_j\big(y_j(t) - c^T\mathbf{x}_j(t) + d\big) \\
+ \big[\omega_j(t) + g_j(x_i(t))\big]^+\nabla_{x_i}g_j\big(x_j(t)\big)\Big).
\end{aligned}
$$

Each $\dot{x}_j(t)$ can be viewed as an input of linear system (8). Since graph $\mathcal{G}(\mathcal{A})$ is connected, by [49, Lemma 3], we know $(L_j + \Lambda_j)$ is positive definite; thus, $\underline{\lambda}_j > 0$ for $j \in \mathcal{V}$. Letting $\lambda = \max_{1\leq j\leq n} \exp(-\underline{\lambda}_j)$ and $K = \max_{1\leq j\leq n} K_j$, it is obvious that $0 < \lambda < 1$. By (8), it follows:

$$
\|e_j\| \leq \lambda^t\|e_j(0)\| + 2K\int_0^t \alpha_\tau \lambda^{(t-\tau)} d_\tau. \tag{10}
$$

Since $\lim_{t\to\infty} \alpha_t = 0$, by applying Lemma 3 to (10), it can be obtained that $\lim_{t\to\infty} e_j = 0$.

2) Denote the Laplacian matrix of $\mathcal{G}(\mathcal{A})$ by $L = [l_{ik}]_{n\times n}$, where $l_{ik} = -a_{ik}$ if $i \neq k$ and $l_{ik} = \sum_{k\in N_i} a_{ik}$ if $i = k$, since $\mathcal{G}(\mathcal{A})$ is connected, the n eigenvalues of L can be given as $0 = \lambda_1(L) < \lambda_2(L) \leq \cdots \leq \lambda_n(L)$. See [6] for details. Letting $y = [y_1, \ldots, y_n]^T$ and $\bar{y} = y - \mathbf{1}_n Y$, we have

$$
\dot{\bar{y}} = -L\bar{y} - \left(I_n - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\right)\phi(t) \tag{11}
$$

where $\phi(t) = (n\alpha_t/2)[c_1{}^T x_1(t) - d/n, \ldots, c_n{}^T x_n(t) - d/n]^T$. Since L is symmetric for $\mathcal{G}(\mathcal{A})$ being undirected, there must exist an orthogonal matrix $Q = [(\mathbf{1}_n/\sqrt{n}), Q_0]$ such that $Q^T L Q = \mathrm{diag}(0, \lambda_2(L), \ldots, \lambda_n(L))$. Note $Q^T(I_n - [\mathbf{1}_n\mathbf{1}_n^T/n]) = [0_{n\times 1}, Q_1^T]^T$, where $Q_1 = Q_0^T(I_n - [\mathbf{1}_n\mathbf{1}_n^T/n])$, from (11), it follows:

$$
Q^T\dot{\bar{y}} = \mathrm{diag}(0, \lambda_2(L), \ldots, \lambda_n(L))Q^T\bar{y} - [0_{n\times 1}, Q_1^T]^T\phi.
$$

Based on above dynamics, together with the fact $\|Q\| = 1$, we can conclude

$$
\begin{aligned}
\|\bar{y}\| &= \|QQ^T\dot{\bar{y}}\| \\
&\leq \|Q^T\dot{\bar{y}}\| \\
&\leq \vartheta^t\|\bar{y}(0)\| + \zeta\Big\|\int_0^t \phi(\tau)\vartheta^{(t-\tau)} d_\tau\Big\|
\end{aligned}
$$

where $\vartheta = \exp(-\lambda_2(L))$ and $\zeta = \|[0_{n\times 1}, Q_1^T]\|$. Recalling the fact that $x_j \in \Omega_j$ and the boundedness of Ω_j, we know $\|c_j{}^T x_j(t) - d/n\|$ is bounded for any $j \in \mathcal{V}$, which implies $\lim_{t\to\infty}\phi(t) = 0$. By the fact $0 < \vartheta < 1$ and reusing Lemma 3, we can conclude $\lim_{t\to\infty}\|\bar{y}\| = 0$.

3) Due to the fact that α_t is nonincreasing, from (10), one has

$$\begin{aligned}\int_0^\infty \alpha_t\|e_j\|d_t &\le \alpha_0\|e_j(0)\|\int_0^\infty \lambda^t d_t\\ &\quad + 2K\int_0^\infty \alpha_t\int_0^t \alpha_\tau\lambda^{t-\tau}d_\tau d_t\\ &= \frac{\alpha_0\|e_j(0)\|}{-\ln\lambda} + 2K\int_0^\infty\int_0^t \alpha_t\alpha_\tau\lambda^{t-\tau}d_\tau d_t.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\int_0^\infty\int_0^t \alpha_t\alpha_\tau\lambda^{t-\tau}d_\tau d_t &= \int_0^\infty\int_0^t \lambda^\theta\alpha_t\alpha_{t-\theta}d_\theta d_t\\ &= \int_0^\infty \lambda^\theta\int_\theta^\infty \alpha_t\alpha_{t-\theta}d_t d_\theta\\ &\le \int_0^\infty \lambda^\theta\int_\theta^\infty \alpha_{t-\theta}^2 d_t d_\theta\\ &\le \frac{1}{-\ln\lambda}\int_0^\infty \alpha_t^2 d_t\\ &< \infty\end{aligned} \tag{12}$$

where the first equality holds by letting $t-\tau = \theta$, and the second one results by changing the order of the integrals. By (12), it holds that $\int_0^\infty \alpha_t\|e_j\|d_t < \infty$. This leads to the validity of the result. ■

Due to the diminishing property of α_t, based on (9), we know $\lim_{t\to\infty}\dot{x}_i(t) = 0$ for any $i \in \mathcal{V}$. Similarly, we can conclude $\lim_{t\to\infty}\dot{y}_i(t) = 0$ and $\lim_{t\to\infty}\dot{\omega}_i(t) = 0$ for any $i \in \mathcal{V}$. Nonetheless, it is not enough to determine whether the actions converge to the GNE. With the help of Lemmas 1–4, next we will present the proof of Theorem 1 in detail.

Proof of Theorem 1: First, letting $X = [x_1^T, \ldots, x_n^T]^T$ and $W = [\omega_1, \ldots, \omega_n]^T$, based on (6), we have

$$\begin{cases}\dot{X} = -\alpha_t X + \alpha_t P_\Omega\Big[X - \Big(\bar{\Psi} - \bar{H}c + \nabla G(X)\big[W + g(X)\big]^+\Big)\Big]\\ \dot{Y} = -\frac{\alpha_t}{2}\big(c^T X - d\big)\\ \dot{W} = -\frac{\alpha_t}{2}W + \frac{\alpha_t}{2}\big[W + g(X)\big]^+\end{cases} \tag{13}$$

where $\bar{\Psi}^T = [\nabla_{x_1}J_1{}^T(\mathbf{x}_1), \ldots, \nabla_{x_n}J_n^T(\mathbf{x}_n)]$, $\bar{H} = \mathrm{diag}((Y - c^T\mathbf{x}_1 + d)I_m, \ldots, (Y - c^T\mathbf{x}_n + d)I_m)$, $\nabla G(X) = \mathrm{diag}(\nabla_{x_1}g_1(x_1), \ldots, \nabla_{x_n}g_n(x_n))$, and $g(X) = [g_1(x_1), \ldots, g_n(x_n)]^T$. Before going on, we define some other variables as $\Psi(X) = [\nabla_{x_1}J_1{}^T(X), \ldots, \nabla_{x_n}J_n^T(X)]^T$, $\bar{X} = P_\Omega[X - (\bar{\Psi} - \bar{H}c + \nabla G(X)[W + g(X)]^+)]$, $\tilde{W} = [W + g(X)]^+$, $Z = [X^T, Y^T, W^T]^T$, and $Z^* = [x^{*T}, y^{*T}, W^{*T}]^T$. Based on (13), we consider a Lyapunov function as follows:

$$V(Z,t) = \varphi(Z) - \varphi\big(Z^*\big) - \big(Z - Z^*\big)^T\nabla_Z\varphi\big(Z^*\big) + \frac{1}{2}\|Z - Z^*\|^2 \tag{14}$$

where

$$\varphi(Z) = \sum_{i=1}^n J_i\big(x_i, x_{-i}^*\big) + \frac{1}{2}\Big(\|\tilde{W}\|^2 + \big|Y - \big(c^T X - d\big)\big|^2\Big). \tag{15}$$

Note $\|\tilde{W}\|^2 = \sum_{i=1}^n([\omega_i + g_i(x_i)]^+)^2$, and

$$\Big(\big[\omega_i + g_i(x_i)\big]^+\Big)^2 = \begin{cases}(\omega_i + g_i(x_i))^2, & \text{if } \omega_i \ge -g_i(x_i)\\ 0, & \text{otherwise.}\end{cases} \tag{16}$$

Letting $\varpi_i(\xi) = \omega_i + g_i(x_i)$, where $\xi = [x_i^T, \omega_i]^T$, note that $\varpi_i(\xi)$ is convex in ξ, together with the fact that for $\varpi_i \ge 0$, ϖ_i^2 is nondecreasing and convex in ϖ_i, it holds that ϖ_i^2 is convex in ξ if $\omega_i \ge -g_i(x_i)$. This implies that $\|\tilde{W}\|^2$ is convex in W and X. Moreover, $|Y - (c^T X - d)|^2$ is convex in X and Y. Then, $\varphi(Z)$ is convex in Z. As a result, one has $V(Z,t) \ge (1/2)\|Z - Z^*\|^2$.

In the following equation, we will prove the convergence of $V(Z,t)$. From (14)–(16), there are $\nabla_W\|\tilde{W}\|^2 = 2\tilde{W}$ and $\nabla_X\|\tilde{W}\|^2 = 2\nabla G(X)\tilde{W}$. Thus, we have

$$\nabla_Z V(Z,t) = \Phi(Z) - \Phi\big(Z^*\big) + Z - Z^* + \Upsilon \tag{17}$$

where

$$\begin{aligned}\Phi(Z) &= \begin{bmatrix}\Psi(X) - c(Y - c^T X + d) + \nabla G(X)\tilde{W}\\ Y - c^T X + d\\ \tilde{W}\end{bmatrix}\\ \Upsilon &= \begin{bmatrix}\tilde{\Psi} - \Psi(X)\\ 0\\ 0\end{bmatrix}\\ \tilde{\Psi} &= \big[\nabla_{x_1}J_1{}^T(x_1, x_{-1}^*), \ldots, \nabla_{x_n}J_n^T(x_n, x_{-n}^*)\big]^T.\end{aligned}$$

Based on (17), taking the derivative of function $V(Z,t)$ with respect to t yields

$$\begin{aligned}\dot{V}(Z,t) &= (\nabla_Z V(Z,t))^T\dot{Z}\\ &= \alpha_t\big(\bar{X} - X\big)^T\Big(X - x^* - \Psi\big(x^*\big) + cy^* - \nabla G(X)W^*\\ &\qquad + \Psi(X) - c\big(Y - c^T X + d\big)\\ &\qquad + \nabla G(X)\tilde{W} + \tilde{\Psi} - \Psi(X)\Big)\\ &\quad + \frac{\alpha_t}{2}|c^T X - d|^2 - \alpha_t\big(Y - y^*\big)^T\big(c^T X - d\big)\\ &\quad - \frac{\alpha_t}{2}\|W - \tilde{W}\|^2 - \alpha_t\big(\tilde{W} - W^*\big)^T\big(W - \tilde{W}\big).\end{aligned}$$

By the fact that $(X - x^*)^T c(Y - c^T X + d) = -|c^T X - d|^2 + Y(c^T X - d)$ and $W - \tilde{W} = [W + g(X)]^- - g(X)$, it follows:

$$\begin{aligned}\dot{V}(Z,t) &= \alpha_t\big(\bar{X} - x^*\big)^T\big(\bar{X} - X + \Psi(X) - c\big(Y - c^T X + d\big)\\ &\quad + \nabla G(X)\tilde{W}\big) + \alpha_t\big(\bar{X} - X\big)^T\Big(\tilde{\Psi} - \Psi(X)\Big)\\ &\quad - \alpha_t\|\bar{X} - X\|^2 - \alpha_t\big(\bar{X} - x^*\big)^T\big(\Psi\big(x^*\big) - cy^*\\ &\quad + \nabla G\big(x^*\big)W^*\big) - \frac{\alpha_t}{2}\big|c^T X - d\big|^2 - \frac{\alpha_t}{2}\|W - \tilde{W}\|^2\\ &\quad - \alpha_t\big(X - x^*\big)^T\big(\Psi(X) - \Psi\big(x^*\big)\big)\\ &\quad - \alpha_t\tilde{W}^T\Big(\big[W + g(X)\big]^- - g(X)\\ &\qquad + (\nabla G(X))^T\big(X - x^*\big)\Big)\\ &\quad + \alpha_t W^{*T}\Big(\big[W + g(X)\big]^- - g(X)\\ &\qquad + \big(\nabla G\big(X^*\big)\big)^T\big(X - x^*\big)\Big).\end{aligned} \tag{18}$$

Noting that $\tilde{W}^T[W+g(X)]^- = 0$, $\tilde{W}^T g(x^*) \le 0$, $W^{*T}[W+g(X)]^- \le 0$ and $W^{*T} g(x^*) = 0$, one has

$$\begin{aligned}&\tilde{W}^T\Big(\big[W+g(X)\big]^- - g(X) + (\nabla G(X))^T\big(X-x^*\big)\Big)\\&\ge \tilde{W}^T\big(g\big(x^*\big) - g(X) + (\nabla G(X))^T\big(X-x^*\big)\big)\\&\ge 0\end{aligned}$$

where the second inequality results from the convexity of g_i. Similarly, we also have

$$W^{*T}\Big(\big[W+g(X)\big]^- - g(X) + (\nabla G(X))^T\big(X-x^*\big)\Big) \ge 0.$$

Moreover, by KKT condition (5), we know $(\bar{X}-x^*)^T(\Psi(x^*) - cY^* + \nabla G(x^*)W^*) \ge 0$, together with the fact that $(X-x^*)^T(\Psi(X) - \Psi(x^*)) \ge \mu\|X-x^*\|^2$, from (18), it follows that:

$$\begin{aligned}\dot{V}(Z,t) \le\ & \alpha_t\big(\bar{X}-x^*\big)^T\big(\bar{X} - X + \Psi(X) - \bar{H}c + \nabla G(X)\tilde{W}\big)\\&+ \alpha_t\big(\bar{X}-x^*\big)^T\big(\bar{H}c - c\big(Y - c^T X + d\big)\big)\\&+ \alpha_t\big(\bar{X}-X\big)^T\Big(\tilde{\Psi} - \Psi(X)\Big)\\&- \alpha_t\|\bar{X}-X\|^2 - \frac{\alpha_t}{2}\big|c^T X - d\big|^2\\&- \frac{\alpha_t}{2}\|W-\tilde{W}\|^2 - \mu\alpha_t\|X-x^*\|^2. \qquad (19)\end{aligned}$$

Since Ω is compact, there must exist positive numbers σ such that $\|\bar{X}-x^*\| \le \sigma$. In the projection inequality (1) of Lemma 1, letting $\mathcal{K} = \Omega$, $u = X - \bar{\Psi} + \bar{H}c - \nabla G(X)\tilde{W}$ and $v = x^*$, we have

$$\begin{aligned}&\big(\bar{X}-x^*\big)^T\big(\bar{X} - X + \Psi(X) - \bar{H}c + \nabla G(X)\tilde{W}\big)\\&= \big(\bar{X}-x^*\big)^T\big(\bar{X} - X + \bar{\Psi} - \bar{H}c + \nabla G(X)\tilde{W}\big)\\&\quad + \big(\bar{X}-x^*\big)^T\big(\Psi(X) - \bar{\Psi}\big)\\&\le \big(\bar{X}-x^*\big)^T\big(\Psi(X) - \bar{\Psi}\big)\\&\le \sigma\|\Psi(X) - \bar{\Psi}\|. \qquad (20)\end{aligned}$$

Furthermore, using Young's inequality, we have

$$\begin{aligned}&\|\bar{X}-X\|\|\tilde{\Psi} - \Psi(X)\|\\&\le \frac{\epsilon}{2}\|\bar{X}-X\|^2 + \frac{1}{2\epsilon}\|\tilde{\Psi} - \Psi(X)\|^2\\&\le \frac{\epsilon}{2}\|\bar{X}-X\|^2 + \frac{(n-1)\gamma^2}{2\epsilon}\|X-x^*\|^2 \qquad (21)\end{aligned}$$

where ϵ is a positive constant such that $([(n-1)\gamma^2]/2\mu) < \epsilon < 2$ and the second inequality results by using the Lipschitz continuity in Assumption 2. Denoting $\rho_1 = 1-(\epsilon/2)$ and $\rho_2 = \mu - ([(n-1)\gamma^2]/2\epsilon)$, it is obvious that $\rho_1, \rho_2 > 0$. Define $e = [e_1^T, \ldots, e_n^T]^T$, where e_i is defined as (8) for any $i \in \mathcal{V}$, and reusing the Lipschitz continuity, we have $\|\Psi(X) - \bar{\Psi}\| \le \gamma\|e\|$. Note $\bar{H}c - c(Y - c^T X + d) \le \|c\|^2\|e\|$, and submitting (20) and (21) to (19) yields

$$\begin{aligned}\dot{V}(Z,t) \le\ & -\rho_1\alpha_t\|\bar{X}-X\|^2 - \frac{\alpha_t}{2}\big|c^T X - d\big|^2 - \frac{\alpha_t}{2}\|W-\tilde{W}\|^2\\&- \rho_2\alpha_t\|X-x^*\|^2 + \rho_3\alpha_t\|e\| \qquad (22)\end{aligned}$$

where $\rho_3 = \sigma(\gamma + \|c\|^2)$. The term $\rho_3\alpha_t\|e\|$ can viewed as a perturbation. By 3) in Lemma 4, we have $\int_0^\infty \alpha_t\|e\|d_t < \infty$. Now, we denote function $h(t) = \rho_3\int_0^t \alpha_s\|e(s)\|d_s$, and it is obvious that $h(t)$ is nondecreasing with respect to t. Together with the fact that $h(t)$ is upper bounded, we know $h(t)$ converges. Integrating both sides of inequality (22) over $[0, t]$ yields

$$\begin{aligned}V(Z(t),t) \le\ & -\rho_1\int_0^t \alpha_s\|\bar{X}-X\|^2 d_s + V(Z(0),0)\\&- \int_0^t \frac{\alpha_s}{2}|c^T X - d|^2 d_s - \int_0^t \frac{\alpha_s}{2}\|W-\tilde{W}\|^2 d_s\\&- \rho_2\int_0^t \alpha_s\|X-x^*\|^2 d_s + h(t)\\&< \infty. \qquad (23)\end{aligned}$$

For any $0 \le t_1 \le t_2$, based on (23), it holds

$$V(Z(t_2),t_2) - V(Z(t_1),t_1) \le h(t_2) - h(t_1)$$

which implies

$$\begin{aligned}&\limsup_{t\to\infty} V(Z(t),t) - \liminf_{t\to\infty} V(Z(t),t)\\&\le \limsup_{t\to\infty} h(t) - \liminf_{t\to\infty} h(t)\\&= 0.\end{aligned}$$

Thus, $V(Z(t),t)$ converges and $V(Z(\infty),\infty)$ exists. Furthermore, from (23), it follows that:

$$\begin{aligned}&\rho_1\int_0^\infty \alpha_s\|\bar{X}-X\|^2 d_s + \int_0^\infty \frac{\alpha_s}{2}\big|c^T X - d\big|^2 d_s\\&+ \int_0^\infty \frac{\alpha_s}{2}\|W-\tilde{W}\|^2 d_s + \rho_2\int_0^\infty \alpha_s\|X-x^*\|^2 d_s\\&< \infty.\end{aligned}$$

Due to the fact $\int_0^\infty \alpha_t d_t \to \infty$, we have

$$\begin{aligned}\liminf_{t\to\infty}\Big(&\rho_1\|\bar{X}-X\|^2 + 1/2\big|c^T X - d\big|^2\\&+ 1/2\|W-\tilde{W}\|^2 + \rho_2\|X-x^*\|^2\Big) = 0.\end{aligned}$$

Hence, there exists a subsequence $\{t_k\}$ such that $\lim_{k\to\infty}\{\|\bar{X}-X\|\}_{t=t_k} = 0$, $\lim_{k\to\infty}|c^T X(t_k) - d| = 0$, $\lim_{k\to\infty}\|W(t_k) - \tilde{W}(t_k)\| = 0$, and $\lim_{k\to\infty}\|X(t_k) - x^*\| = 0$. Thus, KKT condition (3) in Lemma 2 is satisfied. We have

$$\begin{cases}\lim_{k\to\infty} X(t_k) = x^*\\ \lim_{k\to\infty} Y(t_k) = y_0\\ \lim_{k\to\infty} W(t_k) = W_0\end{cases}$$

for some $y_0 \in \mathbb{Y}^*$ and $W_0 \in \mathbb{W}^*$. If we set $y^* = y_0$ and $W^* = W_0$, by the convergence of $V(Z,t)$, we know $\lim_{t\to\infty} V(Z(t),t) = 0$, which implies $\lim_{t\to\infty}\|X-x^*\| = 0$, $\lim_{t\to\infty}|Y-y^*| = 0$, and $\lim_{t\to\infty}\|W-W^*\| = 0$. By 1) and 2) in Lemma 4, we have $\lim_{t\to\infty}\|\mathbf{x}_i - x^*\| = 0$ and $\lim_{t\to\infty}|y_i - y^*| = 0$ for any $i \in \mathcal{V}$. This leads to the validity of the result. ∎

Remark 3: When player i updates the value of $y_i(t)$ by using algorithm (6), the number of agents is utilized. It means that players need to know information associated with the scale of the network, which may prevent the proposed algorithm from being fully distributed. In fact, for a connected undirected multiagent network, it is not difficult to check the value of n by using little global information. For example, we arbitrarily choose an agent's initial state to be 1, and set others' to be 0. Using the average-consensus algorithm in [6], we know all agents' states converge to $1/n$, which helps each one achieve the number of agents. In many practical applications, parameter d can also be propagated to each individual through a connected multiagent network. For example, in the economic dispatch of distributed power system [50], d represents the total load of the power network and it can be measured by each individual in a distributed manner. Moreover, total resource d is usually determined by local demands, that is, $d = \sum_{i=1}^{n} d_i$ [27]. In fact, based on the proof of Theorem 1, we know that the results still hold if $d = \sum_{i=1}^{n} d_i$ and the term d/n in (6) is replaced by d_i. Using local information associated with d_i, rather than d/n, can further reduce the required global information in implementing algorithm (6).

Remark 4: Based on the proof of Theorem 1, here we analyze the convergence rate of algorithm (6) by choosing $\alpha_t = [1/(t+1)]$. To do this, it suffices to analyze the decaying rate of the deviation of actions from the GNE. We use a weighted squared error to reflect this, which is defined as

$$D(t) = \frac{\int_0^t \alpha_\tau \|X(\tau) - x^*\|^2 d_\tau}{\int_0^t \alpha_\tau d_\tau}.$$

Based on (10) and (12), there is

$$\begin{aligned}\int_0^t \alpha_\tau \|e(\tau)\| d_\tau &\leq n^2 m\alpha_0 \max_{1\leq i\leq n} \|e_i(0)\| \int_0^t \lambda^\tau \\ &\quad + 2n^2 mK \int_0^t \alpha_\tau \int_0^\tau \alpha_\theta \lambda^{(\tau-\theta)} d_\theta d_\tau \\ &\leq \frac{n^2 m\big(\alpha_0 \max_{1\leq i\leq n} \|e_i(0)\| + 2K\big)}{-\ln\lambda} \end{aligned} \tag{24}$$

where K and λ are defined in (10). By (24), we have $h(t) \leq \ell$. Let $\ell = ([n^2 m\rho_3(\alpha_0 \max_{1\leq i\leq n} \|e_i(0)\| + 2K)]/-\ln\lambda)$; on the basis of (23), we can conclude that

$$\begin{aligned} D(t) &\leq \frac{V(Z(0), 0) + h(t)}{\rho_2 \ln(t+1)} \\ &\leq \frac{V(Z(0), 0) + \ell}{\rho_2 \ln(t+1)} \end{aligned} \tag{25}$$

which implies that the convergence rate of algorithm (6) is proportional to $1/\ln(t+1)$.

IV. Simulations

In this section, we consider a game with six players to illustrate the obtained results, the index set of players is denoted by $\mathcal{V} = \{1, \ldots, 6\}$. The players communicate with each other via a connected graph depicted in Fig. 1, where the weight of each edge is set to be one. Players in the game intend to selfishly minimize their own cost function subject to some constraints. Each player's cost function in the game is given by

$$J_i = T_{ii}\|x_i\|^2 + \sum_{j=1, j\neq i}^{6} T_{ij} x_i^T x_j, \quad i \in \mathcal{V}$$

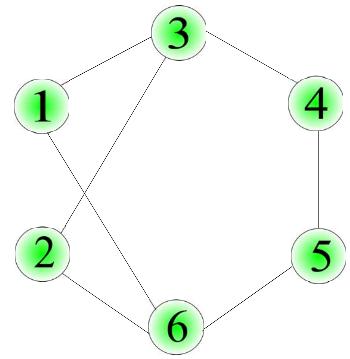

Fig. 1. Communication graph $\mathcal{G}(\mathcal{A})$.

where $T = (T_{ij})_{6\times 6}$

$$T = 10^{-3} \times \begin{bmatrix} 75 & 82 & -65 & 38 & -56 & 42 \\ -82 & 67 & -15 & 72 & 26 & -18 \\ 65 & 15 & 80 & 53 & -64 & 43 \\ -38 & -72 & -53 & 72 & -92 & 28 \\ 56 & -26 & 64 & 92 & 60 & -19 \\ -42 & 18 & -43 & -28 & 19 & 70 \end{bmatrix}$$

and $x_i = [x_1^i, x_2^i]^T \in \mathbb{R}^2$. Noting that T is an antisymmetric matrix, it is not difficult to verify that the conditions associated with strong monotonicity and Lipschitz continuity in Assumption 2 are satisfied. And compact action sets are given by $\Omega_1 = \{0 \leq x_1^1 \leq 3, 1 \leq x_2^1 \leq 2\}$, $\Omega_2 = \{1 \leq x_1^2 \leq 4, 0 \leq x_2^2 \leq 2\}$, $\Omega_3 = \{0.5 \leq x_1^3 \leq 2.5, 3 \leq x_2^3 \leq 4\}$, $\Omega_4 = \{0 \leq x_1^4 \leq 1, 0.5 \leq x_2^4 \leq 1.5\}$, $\Omega_5 = \{1.5 \leq x_1^5 \leq 3, 0.5 \leq x_2^5 \leq 2\}$, and $\Omega_6 = \{0 \leq x_1^6 \leq 4, 1 \leq x_2^6 \leq 6\}$. The coupled constraint is given by an equation $h = cx - d = 0$ and private constraints are given by $g_i = a_i^T x_i - b_i \leq 0$, $i = 1, \ldots, 6$. To ensure the solutions in $\Omega_1 \times \cdots \times \Omega_n$ to be nonempty, we choose the following parameters: $c = 10^{-2} \times [4, -10, 2, 2, -4, 2, 6, -2, 4, -2, -6, 24]^T$ and $d = 0.26$; $a_1 = [10, -30]^T$, $a_2 = [-40, 10]^T$, $a_3 = [20, -5]^T$, $a_4 = [10, -10]^T$, $a_5 = [20, -10]^T$, $a_6 = [0, 10]^T$, $b_1 = b_3 = -10$, $b_2 = 10$, $b_4 = b_5 = b_6 = 20$. It is obvious that each g_i is linear; thus, it satisfies the condition associated with the convexity of g_i in Assumption 1. Together with the fact that J_i is convex in x_i, and Ω_i is convex and compact for any $i \in \mathcal{V}$, the conditions in Assumption 1 are satisfied. Hence, under the aforementioned parameters, all conditions in Assumption 1–3 are satisfied, which means the conditions in Theorem 1 are satisfied. In the following text, we aim to verify that if players make decisions by implementing a distributed algorithm (6), then their actions converge to a GNE of the game eventually.

Algorithm (6) is applied to searching a GNE located in the action sets. We set $\alpha_t = [1/(0.07t + 0.01)]$, which satisfies (7). Let the initial states be $x_1(0) = [0.5, 1.5]^T$, $x_2(0) = [1.5, 1]^T$, $x_3(0) = [1, 3.5]^T$, $x_4(0) = [1, 1.5]^T$, $x_5(0) = [2, 1]^T$, $x_6(0) = [1, 1.5]^T$, $y_1 = 0.4$, $y_2 = 0.5$, $y_3 = 0.7$, $y_4 = 0.8$, $y_5 = 0.2$, $y_6 = -0.4$, and $\omega_i(0) = 0$ and $x_{ij}(0)$, $\forall i \neq j, i, j \in \mathcal{V}$ is randomly chosen from 0 to 1. For $i \in \mathcal{V}$, we denote

$$\begin{aligned} Q_i(t) &= \|x_i - P_{\Omega_i}\big[x_i(t) - \big(\nabla_{x_i} J_i(\mathbf{x}_i(t)) \\ &\quad - Y(t)c_i + \omega_i(t)a_i\big)\big]\| \\ R_i(t) &= \omega_i(t) - \big[\omega_i(t) + a_i^T x_i(t) - b_i\big]^+ \end{aligned}$$

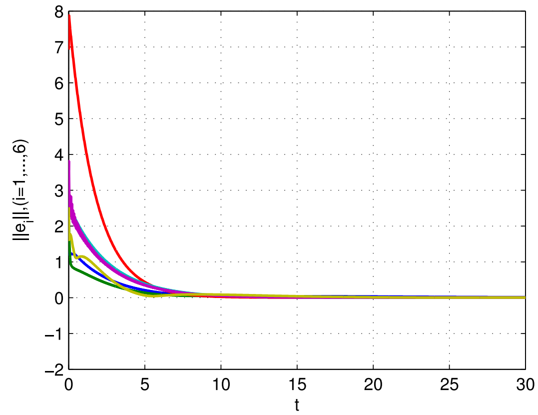

Fig. 2. Trajectories of $\|e_i(t)\|$, $i = 1, \ldots, 6$.

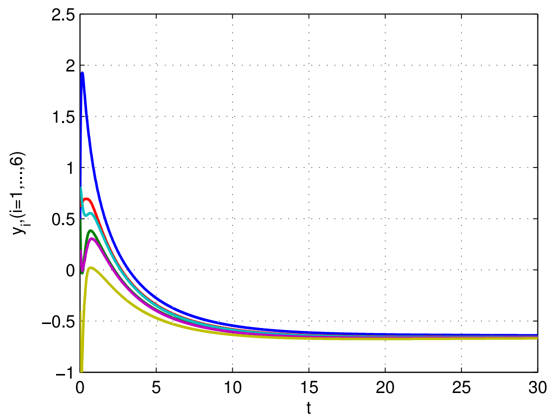

Fig. 3. Trajectories of $y_i(t)$, $i = 1, \ldots, 6$.

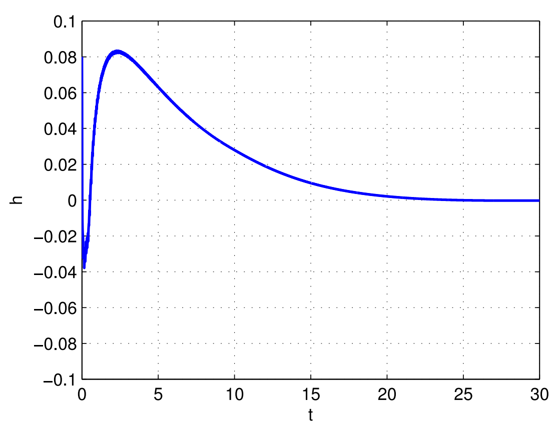

Fig. 4. Trajectory of $h(t)$.

and

$$h(t) = cx(t) - d$$

where $Y(t) = (1/6)\sum_{i=1}^{6} y_i(t)$. Under algorithm (6), the trajectories of $\|e_i\|$, $i \in \mathcal{V}$, defined as (8), are shown in Fig. 2, from which we know as $t \to \infty$, each $\mathbf{x}_i(t)$ defined in (6) asymptotically converges $x(t) = (x_i(t), x_{-i}(t))$. Together with the results in [40] and [41], which are summarized in Lemma 2, we know that $\mathbf{x}_i$ is the GNE if $h = 0$, $Q_i = 0$, and $R_i = 0$ for any $i \in \mathcal{V}$. By (6), the trajectories of players' estimates on the Lagrangian multiplier associated with the coupled equation constraint are shown in Fig. 3, from

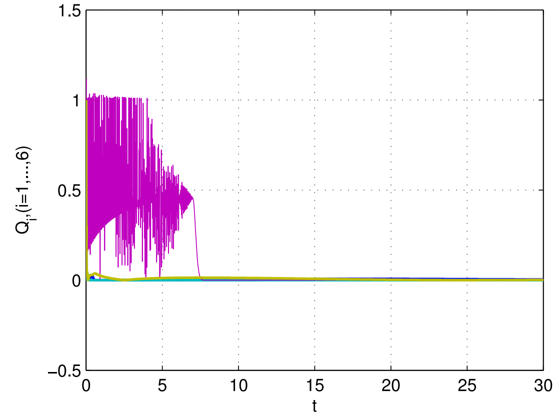

Fig. 5. Trajectories of $Q_i(t)$, $i = 1, \ldots, 6$.

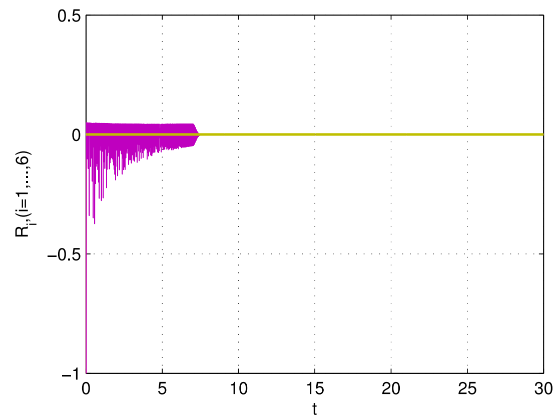

Fig. 6. Trajectories of $R_i(t)$, $i = 1, \ldots, 6$.

which we know they reach a common point after a period of time. Figs. 4–6 show the trajectories of h, Q_i, and R_i, respectively, from which we know $Q_i(t) \to 0$, $h(t) \to 0$, and $R_i(t) \to 0$ after a period of time. Thus, $(x_1^T, \ldots, x_6^T)$ converges to a GNE $(x_1^{*T}, \ldots, x_6^{*T})$ of the game. It is computed that $x_1^* = [0, 1.09]^T$, $x_2^* = [1, 0.71]^T$, $x_3^* = [0.5, 4]^T$, $x_4^* = [1, 1.5]^T$, $x_5^* = [1.5, 1]^T$, and $x_6^* = [0.92, 1.08]^T$. These observations are consistent with the results established in Theorem 1.

V. Conclusion

In this paper, we have presented a continuous-time DGP algorithm for searching a GNE of noncooperative games with feasible action sets constrained by private inequalities and coupled equations. By implementing the algorithm, each player makes decisions by using the information associated with its own cost function, its own private constraint and the coupled constraint, its own action, and actions and estimates received from its neighbors. The convergence of the continuous-time nonlinear DGP algorithm has been proved by using convex analysis theory, consensus theory, and Lyapunov stability theory. The result shows that if the communication graph is connected by the proposed continuous-time DGP algorithm, all players' actions asymptotically converge to a GNE of the game. A simulation example has been conducted to demonstrate the effectiveness of our results. How to completely avoid using the global information and how to reduce the communication costs are two difficult problems in searching a GNE of noncooperative games over a network of players, which will be considered in our future work.

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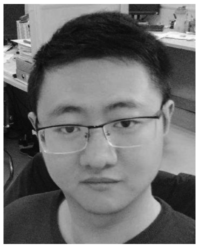

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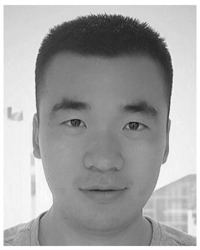

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