Finite-Time Coordination Under State-Dependent Communication Graphs With Inherent Links

Gangshan Jing and Long Wang

***Abstract*—This brief investigates distributed coordination control problems under a state-dependent communication graph with several inherent links. In the network, an interaction arises between two agents if either their states differ by less than a fixed range or an inherent communication link exists between them. By considering that each agent has a state-dependent control gain, a nonlinear consensus protocol is proposed for single-integrator modeled multi-agent systems. With the validity of some initial condition, the communication graph is shown to be connected at any time, which guarantees asymptotic consensus. It is further shown that this result can apply to the opinion consensus problem. This brief also provides a novel approach to finite-time coordination protocols synthesis. By simply multiplying a time-varying gain, any coordination protocol can be transformed into a finite-time protocol. A simulation example is presented to illustrate the effectiveness of the theoretical result.**

***Index Terms*—Consensus, state-dependent graphs, opinion dynamics, formation control, finite-time convergence.**

I. Introduction

Decentralized coordination control of multi-agent systems (MASs) has received considerable attention [1]–[4]. Among those popular issues associated with coordination control, reaching consensus is a fundamental problem for accomplishing more complex tasks. Consensus is said to be reached if a group of agents agree upon specified quantities. The quantities might be position [1], velocity [2], temperature [3], or opinion [5]–[8]. To date, a lot of experimental and theoretical efforts have been carried out on consensus problems [9]–[14]. These works show that the network connectivity plays a pivotal role for reaching consensus. However, in practical applications, the communication graph and the communication data transmission rate usually change as the evolution of agents' states [1], [15]–[17]. As a result, it becomes a critical issue that how to determine the connectivity of the state-dependent communication graph. Related investigations can be found in [1], [5]–[7], and [16].

Manuscript received September 29, 2017; revised May 31, 2018 and July 10, 2018; accepted August 15, 2018. Date of publication August 20, 2018; date of current version May 28, 2019. This work was supported by the National Science Foundation of China under Grant 61533001 and Grant 61751301. This brief was recommended by Associate Editor M. Cao. *(Corresponding author: Long Wang.)*

G. Jing is with the Center for Complex Systems, School of Mechano-electronic Engineering, Xidian University, Xi'an 710071, China (e-mail: nameisjing@gmail.com).

L. Wang is with the Center for Systems and Control, College of Engineering, Peking University, Beijing 100871, China (e-mail: longwang@pku.edu.cn).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TCSII.2018.2866052

In this brief, we consider that any two agents interact with each other if either their states differ by less than a fixed range or an inherent communication link exists between them. That is, the communication graph is the union of a state-dependent graph and a fixed graph. Two examples can be given to show that our assumption on the graph is well-justified. First, during an opinion formation process, besides similar opinions, the inherent relationship between two individuals may also lead to their communications. It is common to observe that people are often influenced by their relatives even if they have totally different opinions. Second, in cooperative control of multiple mobile robots with limited visibility, in addition to interactions between robots located within the sensing range, an inherent information transmission channel may also exist between two robots if they are both equipped with extra wireless sensors.

Considering the fact that different agents may have different ability to utilize the measured information, each agent has a state-dependent gain in our model. Reference [3] has also studied a similar issue under assumptions that the communication graph is connected and each agent is with one-dimensional state. We consider the case when the communication graph has state-dependent connectivity. Moreover, our protocols can apply to agents with multi-dimensional states.

In the literature, a potential function is usually employed to preserve network connectivity [1], [2]. However, when a potential force is applied, the trajectory of each agent will be influenced, which results in a different stable state. This is often unavailable in many cases. As an example, the opinion formation models in [5]–[8] mainly study the evolution of opinions under specified updating laws, applying a potential force will evidently change the original properties of the model. Motivated by this fact, we develop in this brief an approach to predicting network connectivity by initial conditions without applying extra potential forces. We show that the communication graph will keep connected if an initial condition is satisfied. As a result, consensus can be reached among all agents asymptotically.

To achieve crucial control accuracy, finite-time convergence is often desired. Finite-time stability of a class of autonomous systems has been widely studied in the references, e.g., [10], [18]. In their investigations, only the upper bound of convergence time can be computed and the accurate convergence time cannot be given. Yong *et al.* [12] proposed some protocols to make a group of agents reach consensus at a preset time. Nevertheless, their analysis approach only applies to specific linear systems. In this brief, we provide a novel approach for achieving convergence at a preset time. We prove that by employing an artificial time-varying control gain, an autonomous system with locally asymptotically stable equilibria can be modified such that the solution of the modified system converges into the previous equilibrium set at a preset time. By utilizing this method, we propose a finite-time consensus protocol for MASs with single-integrator dynamics. It is important to note that with given initial conditions for asymptotic consensus protocol, the finite-time consensus protocol finally drives the agents to achieve the same consensus state as the asymptotic consensus protocol does.

Notations: Throughout this brief, $\mathbb{R}$ denotes the set of real numbers; $\mathbb{R}^n$ is the $n-$dimensional Euclidean space; $\otimes$ is the kronecker product; $||\cdot||$ stands for the Euclidean norm; C_n^k represents the number of $k-$combiniations from a given set of n elements, *i.e.*, $C_n^k = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1}$; $f'(t) = \frac{df}{dt}$ is the derivative of f with respect to t; $\mathbf{1}_n$ is a vector with n components, where each component is 1.

The interaction between all agents is denoted by a weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. $\mathcal{V} = \{1, \ldots, n\}$ denotes a set of vertices, $\mathcal{E}$ is a set of edges, $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is the adjacency matrix, where a_{ij} is the transmission weight between agents i and j. The neighbor set of i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} | (i,j) \in \mathcal{E}\}$. We always consider that $a_{ij} \geq 0$ and $a_{ii} = 0$ for any $i, j \in \mathcal{V}$, $(j, i) \in \mathcal{E}$ if and only if $a_{ij} > 0$. We say agents i and j are adjacent in $\mathcal{G}$ if $(i, j) \in \mathcal{E}$ and non-adjacent otherwise. The Laplacian matrix of graph $\mathcal{G}$ is defined by $\mathcal{L} = [l_{ij}]$, where $l_{ij} = -a_{ij}$ if $i \neq j$, $l_{ii} = \sum_{k\in\mathcal{N}_i} a_{ik}$. A connected subgraph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \mathcal{A}')$ of $\mathcal{G}$ is called a connected component if for any $i \in \mathcal{V}'$, $j \in \mathcal{V}$ and $(i, j) \in \mathcal{E}$, it always holds that $j \in \mathcal{V}'$ and $(i, j) \in \mathcal{E}'$. The connected component with minimum number of vertices is called the minimum connected component.

II. Problem Formulation

A. State-Dependent Communication Graphs With Inherent Links

In the literature, connectivity of the network is often guaranteed by using artificial potential functions [1], [2] or restricting agents' initial states [7], [16]. However, many system models characterizing social networks and physical phenomena cannot be controlled by specific external forces. Moreover, the restriction on initial configurations is often strict and difficult to satisfy. In this subsection, we introduce a communication graph consisting of a state-dependent graph and a time-invariant inherent graph, under which we will show that the restriction on agents' initial states for preserving connectivity can be relaxed comparing to the entirely state-dependent interaction graph in [7] and [16].

In our investigation, the communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is determined by the following two graphs:

(i) A state-dependent graph ${}^{st}\mathcal{G} = (\mathcal{V}, {}^{st}\mathcal{E}, {}^{st}\mathcal{A})$, in which an interaction link between two agents exists if and only if their relative state is less than a specified constant, which is called the sensing radius. For agents i and j, we regard the interaction weight between them as a function of $||x_i - x_j||$, *i.e.,* ${}^{st}a_{ij}(x) = \alpha(||x_i - x_j||)$, where $\alpha(\cdot)$ satisfies the following assumption.

Assumption 1: $\alpha(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous, $0 < \alpha(s) < \infty$ if $s < R$ and $\alpha(s) = 0$ otherwise, where $R > 0$ is a constant.

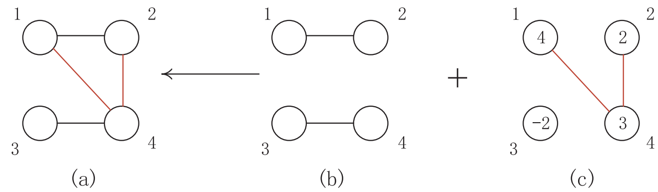

Fig. 1. The communication graph between four agents with states $\{4, 2, -2, 3\}$, the sensing radius is set by 1.5. (a) The communication graph $\mathcal{G}$, (b) the inherent graph, (c) the state-dependent graph.

(ii) An inherent fixed graph ${}^{in}\mathcal{G} = (\mathcal{V}, {}^{in}\mathcal{E}, {}^{in}\mathcal{A})$, in which the weight ${}^{in}a_{ij}$ is a positive constant if $(i, j) \in {}^{in}\mathcal{E}$, and ${}^{in}a_{ij} = 0$ otherwise. We always consider that ${}^{in}\mathcal{G}$ is not connected throughout this brief. Note that in the case when ${}^{in}\mathcal{G}$ is connected, $\mathcal{G}$ is certainly connected at any time. Since our condition for agents' initial states is only for connectivity preservation, the initial condition will be unnecessary and the coordination control strategy can be effective, irrespective of agents' initial states. Throughout this brief, we use n^* to denote the number of vertices in the minimum connected component of ${}^{in}\mathcal{G}$.

More precisely, we have $\mathcal{E} = {}^{st}\mathcal{E} \bigcup {}^{in}\mathcal{E}$ and $\mathcal{A} = {}^{st}\mathcal{A} + {}^{in}\mathcal{A}$. Fig. 1 shows an example of the communication graph considered in this brief. It is easy to see that both ${}^{st}\mathcal{G}$ and ${}^{in}\mathcal{G}$ are not connected, but their union graph $\mathcal{G}$ is connected.

B. System Model and Finite-Time Consensus

We consider agents with singe-integrator dynamics

$$\dot{x}_i = u_i, \qquad i \in \mathcal{V} \tag{1}$$

respectively, where $x_i, u_i \in \mathbb{R}^m$, $\mathcal{V} = \{1, \ldots, n\}$. In the rest of this brief, we adopt $x = (x_1^T, \ldots, x_n^T)^T$ and $u = (u_1^T, \ldots, u_n^T)^T$.

The objective of this brief is to design distributed protocols that can drive a group of agents to reach agreement on their states. The formal definitions of consensus and finite-time consensus are characterized, respectively, as follows.

Definition 1: The MAS (1) is said to reach consensus asymptotically if $\lim_{t\to\infty} ||x_i - x_j|| = 0$ for any $i, j \in \mathcal{V}$.

Definition 2: The MAS (1) is said to reach consensus in finite time if there exists a time $T > 0$, such that $\lim_{t\to T^-} ||x_i - x_j|| \to 0$ for any $i, j \in \mathcal{V}$.

If all agents reach consensus eventually, the common state which all agents converge to is called the consensus state.

III. Main Results

In this section, we will focus on seeking an appropriate initial condition to fix the connectivity and in turn drive the agents to achieve consensus. Before showing the main result, we present the following lemmas, which are critical to the theoretical proofs.

Lemma 1 [19]: Suppose that $\mathscr{F}(\cdot) : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function with nonzero derivative at the point p, then $\mathscr{F}$ is invertible in a neighborhood of p, the inverse is continuously differentiable, and $(\mathscr{F}^{-1})'(\mathscr{F}(p)) = \frac{1}{\mathscr{F}'(p)}$.

Lemma 2: If there are less than $n^*(n - n^*)$ pairs of non-adjacent vertices in graph $\mathcal{G}$, then $\mathcal{G}$ is connected.

Proof: Let $\mathcal{G}(r)$ be a graph having $r \geq 2$ connected components, n_i is the number of agents corresponding to the $i-$th connected component, and $n_1 \leq n_2 \leq \cdots \leq n_r$. Let $g(r)$ be the maximum number of adjacent vertex pairs that $\mathcal{G}(r)$ may have. Then it must hold $g(r) = \sum_{i=p}^{r} C_{n_i}^2$, where $p = \min\{i|n_i \geq 2\}$. Note that

$$\begin{aligned} g(r-1) &\geq C_{n_p+n_{p+1}}^2 + \sum_{i=p+2}^{r} C_{n_i}^2 \\ &\geq C_{n_p}^2 + C_{n_{p+1}}^2 + \sum_{i=p+2}^{r} C_{n_i}^2 = \sum_{i=p}^{r} C_{n_i}^2 = g(r). \end{aligned}$$

Therefore, $g(r)$ is nonincreasing of r, implying that the more connected components a graph has, the smaller the maximum number of adjacent vertex pairs is.

Now we prove that $\mathcal{G}$ is connected by reduction to absurdity. Suppose that $\mathcal{G}$ is not connected. That is, $r \geq 2$. Let $f(r)$ be the number of pairs of non-adjacent agents for graph $\mathcal{G}$. It follows that

$$f(r) \geq C_n^2 - g(r) \geq C_n^2 - g(2).$$

Note that $g(2) = C_{n_p}^2 + C_{n-n_p}^2 = \frac{n^2-n}{2} + n_p^2 - n_p n$. Since the edges of graph ${}^{in}\mathcal{G}$ can never be lost, we have $n^* \leq n_p \leq n-n^*$. It follows that $f(r) \geq C_n^2 - g(2) \geq n_p n - n_p^2 \geq n^*(n-n^*)$. This conflicts with our condition. In conclusion, we obtain that $\mathcal{G}$ is connected. ∎

A. Reaching Asymptotic Consensus

For agents with dynamics (1), the following consensus protocol is considered.

$$u_i = \frac{1}{w_i(||x_i||)} \sum_{j\in\mathcal{V}} a_{ij}(||x_i - x_j||)(x_j - x_i), \quad i \in \mathcal{V}. \tag{2}$$

Assumption 2: $w_i(\cdot) : \mathbb{R} \to \mathbb{R}_{>0}$ is continuous and there exist $\underline{w}$ and $\bar{w}$ such that $0 < \underline{w} \leq w_i \leq \bar{w}$.

In protocol (2), $1/w_i$ is a nonlinear gain depending on the state of agent i. When $x_i \in \mathbb{R}$ for any $i \in \mathcal{V}$, $w_i(||x_i||)$ can be replaced with $w_i(x_i)$ where $w_i(\cdot)$ satisfies Assumption 2. In reality, if x_i denotes the temperature of room i, then $w_i(x_i)$ can be considered as the temperature-dependent heat capacity of room i, see [3].

Theorem 1: Consider an MAS consisting of n agents with dynamics (1) and protocol (2), where $\alpha(\cdot)$ and $w_i(\cdot)$ satisfy Assumption 1 and Assumption 2, respectively. Suppose the following inequality holds:

$$\frac{1}{2}\sum_{i,j\in\mathcal{V}} \int_0^{||x_i(0)-x_j(0)||} s a_{ij}(s)ds < n^*(n-n^*) \int_0^R s\alpha(s)ds, \tag{3}$$

then the consensus problem is solved asymptotically.

Proof: Consider the following Lyapunov candidate:

$$V = \frac{1}{2}\sum_{i,j\in\mathcal{V}} \int_0^{||x_i-x_j||} s a_{ij}(s)ds. \tag{4}$$

Then

$$\begin{aligned} \dot{V} &= \frac{1}{2}\sum_{i,j\in\mathcal{V}} a_{ij}(||x_i - x_j||)(x_i - x_j)^T(\dot{x}_i - \dot{x}_j) \\ &= -\sum_{i\in\mathcal{V}} w_i||\dot{x}_i||^2 \leq 0. \end{aligned} \tag{5}$$

Next we show the compactness of $\Omega = \{x|V(x(t)) \leq V(x(0))\}$. Since Ω is obviously closed, we only prove that it is also bounded. Consider the function $E_1(x) = \sum_{i\in\mathcal{V}} \int_0^{||x_i||} s w_i(s)ds$. Differentiating E_1 along the trajectories of the agents, we have

$$\dot{E}_1 = \sum_{i\in\mathcal{V}} ||x_i|| \cdot \frac{x_i^T \dot{x}_i}{||x_i||} = -\frac{1}{2}\sum_{i,j\in\mathcal{V}} a_{ij}||x_i - x_j||^2 \leq 0.$$

It follows that $\frac{1}{2}\sum_{i\in\mathcal{V}} \underline{w}||x_i(t)||^2 \leq E_1(x(t)) \leq E_1(x(0)) \leq \frac{1}{2}\sum_{i\in\mathcal{V}} \bar{w}||x_i(0)||^2$.

Hence, $||x(t)|| \leq \frac{\bar{w}}{\underline{w}}||x(0)||$, implying the compactness of Ω. By LaSalle's invariance principle, the states of all agents asymptotically converge into the largest invariant set in $M = \{x|\dot{V}(x) = 0\} = \{x|\dot{x} = 0\}$. Let $W = \lim_{t\to\infty} \operatorname{diag}(w_1(||x_1(t)||), \ldots, w_n(||x_n(t)||))$, Assumption 2 implies that W is invertible. Then $M = \{x|W^{-1}(\mathcal{L} \otimes I_m)x = 0\} = \{x|(\mathcal{L} \otimes I_m)x = 0\}$. From (3), one has

$$\begin{aligned} \frac{1}{2}\sum_{i,j\in\mathcal{V}} \int_0^{||x_i(t)-x_j(t)||} s\alpha(s)ds &\leq V(t) \\ &\leq V(0) < n^*(n-n^*) \int_0^R s\alpha(s)ds. \end{aligned}$$

Note that $\int_0^l s\alpha(s)ds$ is increasing of l. As a result, for any agents i and j, $||x_i - x_j|| \geq R$ if and only if $\int_0^{||x_i-x_j||} s\alpha(s)ds = \int_0^R s\alpha(s)ds$. It follows that there are less than $n^*(n-n^*)$ pairs of non-adjacent agents in ${}^{st}\mathcal{G}(t)$. Since $\mathcal{G}(t)$ has more edges than ${}^{st}\mathcal{G}(t)$, $\mathcal{G}(t)$ also has less than $n^*(n-n^*)$ pairs of non-adjacent agents. By lemma 2, $\mathcal{G}(t)$ is connected for any $t \geq 0$. Consequently, the largest invariant set in M is $\operatorname{span}\{\mathbf{1}_n \otimes I_m\}$. Therefore, $\lim_{t\to\infty} ||x_i - x_j|| = 0$ for any $i, j \in \mathcal{V}$. ∎

In particular, if $x_i \in \mathbb{R}$, it is easy to see that Theorem 1 also holds if $w_i(||x_i||)$ in protocol (2) is replaced with $w_i(x_i)$. In this case, we can determine the consensus state by using the function $E_2 = \sum_{i\in\mathcal{V}} \int_0^{x_i} w_i(s)ds$. Note that $\dot{E}_2 = \sum_{i\in\mathcal{V}} \sum_{j\in\mathcal{V}} a_{ij}(||x_i - x_j||)(x_j - x_i) = 0$. Suppose $\lim_{t\to\infty} x_i(t) = x^*$ for all $i \in \mathcal{V}$, it is clear that x^* satisfies $\sum_{i\in\mathcal{V}} \int_0^{x^*} w_i(s)ds = \sum_{i\in\mathcal{V}} \int_0^{x_i(0)} w_i(s)ds$.

Remark 1: It can be observed from (3) that the smaller ${}^{in}a_{ij}$ for $(i,j) \in {}^{in}\mathcal{E}$ is, the milder condition (3) will be. For example, consider the initial state vector $x(0) = (c_1\mathbf{1}_{n^*}^T, c_2\mathbf{1}_{n-n^*}^T)^T$, where $c_1, c_2 \in \mathbb{R}$. Graph ${}^{in}\mathcal{G}$ has two connected components with $\{1, \ldots, n^*\}$ and $\{n^*+1, \ldots, n\}$ being the corresponding sets of agents, respectively. Let ε be the upper bound of ${}^{in}a_{ij}$. Then the invalidity of condition (3) implies that $n^*(n-n^*)\int_0^{|c_1-c_2|} s(\alpha(s)+\varepsilon)ds \geq n^*(n-n^*)\int_0^R s\alpha(s)ds$, that is, $\int_0^{|c_1-c_2|} s\alpha(s)ds + \frac{1}{2}(c_1-c_2)^2\varepsilon \geq \int_0^R s\alpha(s)ds$. If ε is small enough, we can derive that $|c_1 - c_2| \geq R$, which means that the connectivity will never be guaranteed. That is, the sufficiency condition (3) is also close to a necessity condition in this case. However, if ${}^{in}a_{ij} = 0$ for any $i, j \in \mathcal{V}$, *i.e.*, $n^* = 1$, no inherent communication links exist in the graph, the right hand side of (3) will decrease accordingly. Therefore, when several inherent communication links with small communication weights exist in the graph, condition (3) is more relaxed than the one in [16].

B. Application to Opinion Dynamics

In [5], an opinion formation model (Krause's model) was developed under an interaction graph with state-dependent connectivity. It describes a social phenomenon that several individuals discuss with each other and update their opinions by averaging the opinions of their neighbors. The opinion of each individual is modeled as a real number. Two individuals are neighbors if and only if their opinions differ by less than the confidence bound $R > 0$. Now we further assume that there exist some individuals interacting with each other all the time due to the inherent relationships between them. That is, the communication weight between agents i and j can be denoted by $a_{ij} = {}^{in}a_{ij} + {}^{st}a_{ij}$, where ${}^{in}a_{ij} \geq 0$ is fixed, ${}^{st}a_{ij} = \alpha(|x_i - x_j|)$. Then we have the following generalized continuous-time version of Krause's model:

$$\dot{x}_i = \frac{\sum_{j\in\mathcal{N}_i} a_{ij}(|x_i - x_j|)(x_j - x_i)}{\sum_{j\in\mathcal{N}_i} a_{ij}(|x_i - x_j|)}, i \in \mathcal{V}, \tag{6}$$

where $x_i \in \mathbb{R}$, $\mathcal{V} = \{1 \cdots, n\}$. When ${}^{in}a_{ij} = 0$ for all $i, j \in \mathcal{V}$, the model (6) will reduce to a continuous-time Krause's model. Motivated by [6], the communication weight between agents i and j can be chosen as $\alpha(|x_i - x_j|)$. $\alpha(\cdot)$ is set by

$$\alpha(s) = \begin{cases} 1, & 0 < s < R - \epsilon; \\ \frac{1}{\epsilon}(R - s), & R - \epsilon \leq s < R; \\ 0, & s \geq R, \end{cases} \tag{7}$$

where $0 < \epsilon < R$.

The model (6) is similar to MAS (1) with protocol (2). For (6), although $w_i = \sum_{j\in\mathcal{V}} a_{ij}(|x_i - x_j|)$ is not fully dependent on $|x_i|$, if the right hand side of (6) keeps continuous, by following similar lines to the proof of Theorem 1, we can derive that condition (3) also leads to consensus of the model (6). To show this, we note that (5) also holds for the model (6). That is, once condition (3) holds, we have $V(t) \leq V(0) < n^*(n - n^*)\int_0^R s\alpha(s)ds$ for $t \geq 0$. Then graph $\mathcal{G}(t)$ is always connected for $t \geq 0$. This implies that $\sum_{j\in\mathcal{N}_i} a_{ij}(|x_i - x_j|) \neq 0$ for all $i \in \mathcal{V}$ and $t \geq 0$. As a consequence, the right hand side of (6) is continuous for any $t \geq 0$. Since $x_i \in \mathbb{R}$, by intuitive observation from the dynamic equation (6), $\max_{i\in\mathcal{V}} x_i$ is nonincreasing and $\min_{i\in\mathcal{V}} x_i$ is nondecreasing. Hence $|x(t)|^2 \leq n \max_{i\in\mathcal{V}} |x_i(0)|^2$, which results in the compactness of $\{x|V(x(t)) \leq V(x(0))\}$. According to LaSalle's invariance principle, the states of all agents asymptotically converge into $\{x|\dot{x} = 0\} = \{x|\mathcal{L}x = 0\}$. Since we have shown that $\mathcal{G}(t)$ is connected for any $t \geq 0$, x converges into span$\{\mathbf{1}_n\}$ asymptotically. Therefore, the following corollary holds.

Corollary 1: The opinion formation model (6) achieves consensus asymptotically if condition (3) holds.

C. A Time-Varying Gain for Finite-Time Convergence

In this subsection, by constructing a time-varying gain, we provide a novel approach to finite-time coordination protocols synthesis. Different from [10] and [18], the accurate convergence time is independent of the initial states of the system and will be set artificially. We first introduce a function $z(\cdot) : \mathbb{R} \to \mathbb{R}$ satisfying the following assumption.

Assumption 3: (i). $z(t)$ is continuously differentiable with nonzero derivative on $[0, T)$, where T is a positive constant.

(ii). $z(0) = 0$, $z(t) \to \infty$ as $t \to T^-$.

A direct consequence of Assumption 3 is the invertibility of $z(t)$. To see this, we should note that $z'(t)$ is continuous and nonzero on $[0, T)$. That is, $z'(t) > 0$ or $z'(t) < 0$ for all $t \in [0, T)$. Hence, $z(t)$ is either strictly increasing or decreasing on $[0, T)$. As a result, $z(t)$ is a one to one mapping and invertible. In fact, by condition (ii) in Assumption 3, $z(t)$ is a strictly increasing function.

The following proposition shows that by using the gain $z'(t)$, any autonomous system with locally asymptotically stable equilibria can be modified as a system with solutions converging into the previous equilibrium set at a preset time.

Proposition 1: Suppose the solution of an autonomous system

$$\frac{dx}{dt} = f(x) \tag{8}$$

with initial state $x|_{t=0} = x^*$ asymptotically converges into the equilibrium set E as $t \to \infty$, where $x \in \mathbb{R}^m$, $f : \mathbb{R}^m \to \mathbb{R}^m$ is Lipschitz continuous. If $z(t)$ satisfies Assumption 3, then the solution of system

$$\frac{dx}{dt} = z'(t)f(x) \tag{9}$$

with initial state $x|_{t=0} = x^*$ converges into E as $t \to T^-$.

Proof: Due to Assumption 3 and Lemma 1, $z(t)$ is invertible and for any $t \in [0, T)$, we have $(z^{-1})'(z(t)) = \frac{1}{z'(t)}$. By the chain rule, system (9) can be equivalently written as the following system

$$\frac{dx}{dz} = \frac{dx}{dt} \cdot (z^{-1})'(z(t)) = z'(t)f(x) \cdot (z^{-1})'(z(t)) = f(x). \tag{10}$$

Regarding z as the time variable, we study the solution of system (10) from $z = 0$. Since $z(0) = 0$, we have $x|_{z=0} = x|_{t=0} = x^*$. As a consequence, system (10) has the same equilibrium set as system (8) does. That is, x asymptotically converges into E as $z \to \infty$. Note that the invertibility of $z(t)$ implies that $z(t)$ is a one to one mapping. Hence, x gradually converges into E as $t \to T^-$. ■

Remark 2: When system (9) is transformed into system (10), we can regard this fact as the change of the time variable from t to z. In fact, condition (ii) in Assumption 3 can be more relaxed. Note that we just have to ensure that $x|_{z=z^*} = x|_{t=0} = x^*$ for some $z^* \geq 0$, then x will reach the equilibrium set E as $z \to \infty$. Therefore, once it holds that $z(0) > 0$, by regarding $z = z(0)$ as the initial time of system (10), we can also obtain the validity of Proposition 1. That is, the asymptotic stability of the solution corresponding to system (10) with initial state $x|_{z=z(0)} = x^*$ leads to the

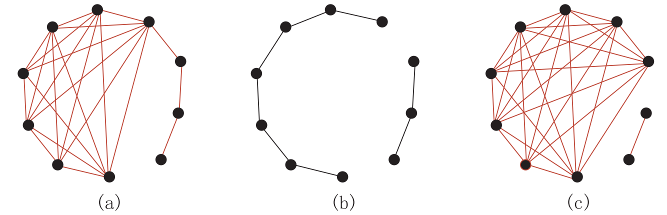

Fig. 2. Three communication graphs.

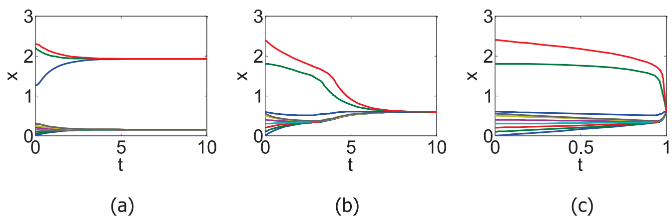

Fig. 3. (a) Consensus is not reached, the agents gradually split into two subgroups. (b) Consensus is reached asymptotically. (c) Consensus is reached at preset time 1.

finite-time convergence of the solution corresponding to system (9) with initial state $x|_{t=0} = x^*$.

D. Reaching Consensus at a Preset Time

According to Assumption 3, a possible choice of $z(t)$ is

$$z(t) = \log T - \log(T - t). \tag{11}$$

In what follows, we will employ this specific time-varying gain to solve the finite-time consensus problem.

It is straightforward that Proposition 1 applies to MAS (1) with protocol (2). Thus we have the following result.

Theorem 2: For an MAS consisting of n agents with dynamics (1), under Assumptions 1 and 2, the protocol

$$u_i = \frac{z'(t)}{w_i(||x_i||)} \sum_{j\in\mathcal{V}} a_{ij}(x_j - x_i), i \in \mathcal{V}, \tag{12}$$

solves the finite-time consensus problem if condition (3) holds.

IV. A Simulation Example

Consider the smoothed opinion formation model (6) with 10 agents interacting with each other. Let $R = 1$ and $\epsilon = 0.01$. The initial opinions of the agents are set by $x(0) = (0, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 1.25, 2.2, 2.3)^T$, which determines the state-dependent communication graph in Fig. 2 (a). With the inherent communication graph in Fig. 2 (b) and ${}^{in}a_{ij} = 1$ for $(i, j) \in {}^{in}\mathcal{E}$, one has $V = 12.01 > 10.4 = 21W$, where V is in form (4), $W = \int_0^R s\alpha(s)ds$. That is, the condition (3) is invalid. Fig. 3 (a) shows that although the initial communication topology is connected, the agents gradually split into two subgroups. Now we set the initial opinions by $x = (0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.55, 0.6, 1.8, 2.4)^T$. Then the state-dependent communication graph becomes the one in Fig. 2 (c). Using the same inherent interaction relationship in Fig. 2 (b), we have $V = 10.37 < 10.4 = 21W$. By Theorem 1, consensus is reached asymptotically, as shown in Fig. 3 (b).

To achieve consensus in finite time, we let $z(t)$ be in form (11), where $T > 0$ is the desired time for reaching consensus. By implementing protocol (12) with $w_i = \sum_{j\in\mathcal{N}_i} a_{ij}$ and setting $T = 1$, Fig. 3 (c) shows trajectories of the opinions under the same initial states as the last example. It can be observed that consensus is reached as $t \to T^-$, and the consensus state is identical to the one in Fig. 3 (b).

V. Conclusion

We solved the consensus problem of continuous-time MASs under state-dependent communication graphs with several inherent communication links. By introducing an artificial time-varying control gain, we proposed a finite-time protocol corresponding to the asymptotic coordination protocol.

References

[1] M. M. Zavlanos and G. J. Pappas, "Potential fields for maintaining connectivity of mobile networks," *IEEE Trans. Robot.*, vol. 23, no. 4, pp. 812–816, Aug. 2007.

[2] G. Jing, Y. Zheng, and L. Wang, "Flocking of multi-agent systems with multiple groups," *Int. J. Control*, vol. 87, no. 12, pp. 2573–2582, 2014.

[3] M. Andreasson, D. V. Dimarogonas, H. Sandberg, and K. H. Johansson, "Distributed control of networked dynamical systems: Static feedback, integral action and consensus," *IEEE Trans. Autom. Control*, vol. 59, no. 7, pp. 1750–1764, Jul. 2014.

[4] Z. Lu, L. Zhang, and L. Wang, "Observability of multi-agent systems with switching topology," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 64, no. 11, pp. 1317–1321, Nov. 2017.

[5] U. Krause, "A discrete nonlinear and non-autonomous model of consensus formation," in *Communications in Difference Equations*. Amsterdam, The Netherlands: Gordon and Breach, 2000, pp. 227–236.

[6] F. Ceragioli and P. Frasca, "Continuous and discontinuous opinion dynamics with bounded confidence," *Nonlin. Anal. Real World Appl.*, vol. 13, no. 3, pp. 1239–1251, 2012.

[7] Y. Yang, D. V. Dimarogonas, and X. Hu, "Opinion consensus of modified Hegselmann–Krause models," *Automatica*, vol. 50, no. 2, pp. 622–627, 2014.

[8] A. V. Proskurnikov, A. S. Matveev, and M. Cao, "Opinion dynamics in social networks with hostile camps: Consensus vs. polarization," *IEEE Trans. Autom. Control*, vol. 61, no. 6, pp. 1524–1536, Jun. 2016.

[9] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[10] L. Wang and F. Xiao, "Finite-time consensus problems for networks of dynamic agents," *IEEE Trans. Autom. Control*, vol. 55, no. 4, pp. 950–955, Apr. 2010.

[11] Y. Zheng and L. Wang, "Consensus of switched multiagent systems," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 63, no. 3, pp. 314–318, Mar. 2016.

[12] C. Yong, X. Guangming, and L. Huiyang, "Reaching consensus at a preset time: Single-integrator dynamics case," in *Proc. 31st Chin. Control Conf. (CCC)*, 2012, pp. 6220–6225.

[13] Y. Zheng, J. Ma, and L. Wang, "Consensus of hybrid multi-agent systems," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 4, pp. 1359–1365, Apr. 2018.

[14] Y. Zhu, S. Li, J. Ma, and Y. Zheng, "Bipartite consensus in networks of agents with antagonistic interactions and quantization," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, to be published, doi: 10.1109/TCSII.2018.2811803.

[15] M. Mesbahi, "On state-dependent dynamic graphs and their controllability properties," *IEEE Trans. Autom. Control*, vol. 50, no. 3, pp. 387–392, Mar. 2005.

[16] G. Jing, Y. Zheng, and L. Wang, "Consensus of multiagent systems with distance-dependent communication networks," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 28, no. 11, pp. 2712–2726, Nov. 2017.

[17] H. Li, G. Chen, T. Huang, and Z. Dong, "High-performance consensus control in networked systems with limited bandwidth communication and time-varying directed topologies," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 28, no. 5, pp. 1043–1054, May 2017.

[18] S. P. Bhat and D. S. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM J. Control Optim.*, vol. 38, no. 3, pp. 751–766, 2000.

[19] W. Rudin, *Principles of Mathematical Analysis*, vol. 3. New York, NY, USA: McGraw-Hill, 1964.