International Journal of Control

ISSN: 0020-7179 (Print) 1366-5820 (Online) Journal homepage: https://www.tandfonline.com/loi/tcon20

Flocking of multi-agent systems with multiple groups

Gangshan Jing, Yuanshi Zheng & Long Wang

To cite this article: Gangshan Jing, Yuanshi Zheng & Long Wang (2014) Flocking of multi-agent systems with multiple groups, International Journal of Control, 87:12, 2573-2582, DOI: 10.1080/00207179.2014.935485

To link to this article: https://doi.org/10.1080/00207179.2014.935485

Published online: 15 Jul 2014.

International Journal of Control, 2014
Vol. 87, No. 12, 2573–2582, http://dx.doi.org/10.1080/00207179.2014.935485

Flocking of multi-agent systems with multiple groups

Gangshan Jing[a,b], Yuanshi Zheng[a,b] and Long Wang[c,*]

[a]*Key Laboratory of Electronic Equipment Structure Design of Ministry of Education, School of Mechano-Electronic Engineering, Xidian University, Xi'an 710071, China;* [b]*Center for Complex Systems, School of Mechano-Electronic Engineering, Xidian University, Xi'an 710071, China;* [c]*Center for Systems and Control, College of Engineering, Peking University, Beijing 100871, China*

(*Received 15 January 2014; accepted 11 June 2014*)

In this paper, we consider the flocking problem of multi-agent systems with multiple groups. First, some algorithms using local information are designed to divide the agents into any pre-assigned number of groups in fixed and switching heterogeneous networks, respectively. Based on algebraic graph theory and Barbalat's lemma, convergence criteria are established to ensure velocity alignment and cohesion of each subgroup as well as collision avoidance between any agents in the whole group. Second, an algorithm for homogeneous networks is studied. Simulation examples are finally presented to verify the effectiveness of our theoretical results.

Keywords: flocking; multiple groups; heterogeneous networks; multi-agent systems

1. Introduction

Over the past several years, the problem of distributed control in multi-agent systems has received increasing attention from different research communities due to its prosperous applications including formation control of multiple air vehicles, synchronisation of self-organised sensor networks, and cooperative control of unmanned vehicles (Akyildiz, Su, Sankarasubramaniam, & Cayirci, 2002; Ji, Wang, Lin, & Wang, 2010; Ren & Beard, 2008; Xiao, Wang, Chen, & Gao, 2009). As an interesting and significant problem derived from the motion of animals in nature such as schools of fish, swarms of bees, migrations of birds (Okubo, 1986; Shaw, 1975), flocking control in multi-agent systems has been investigated for a few years and many meaningful results have been obtained (Cucker & Dong, 2010; Cucker & Smale, 2007; Olfati-Saber, 2006; Reynolds, 1987; Shi, Wang, & Chu, 2006; Tanner, Jadbabaie, & Pappas, 2007; Vicsek, Czirk, Ben-Jacob, Cohen, & Shochet, 1995; Zavlanos, Jadbabaie, & Pappas, 2007).

To date, most researches concerning flocking problems are on the basis of Reynolds' three rules: *separation*, *alignment*, and *cohesion* (Reynolds, 1987). These results focus on the behaviour of agents located in a single group, while neglect the case that agents can move among mixed groups. Actually, phenomena that agents move to different groups are ubiquitously observed in reality. For instance, scattered animals in a system (Zheng & Wang, 2012; Zheng, Zhu, & Wang, 2011) migrate by species, a group of vehicles disperse when facing multi-task, a flock splits when encountering predators, and a community divides into many parties as the incremental conflicts of interests and opinions.

In fact, there has been quite a few results about the division of a group by self-organising on the consensus problems of multi-agent systems (Altafini, 2013; Blondel, Hendrickx, & Tsitsiklis, 2009; Hegselmann & Krause, 2002; Yu & Wang, 2009, 2010). Moreover, several existing investigations on flocking of multi-agent systems have also involved the fragmentation problem of a group. More specifically, Su, Wang, and Yang (2008) studied the case of multiple virtual leaders existing in a flock; Luo, Li, and Guan (2010) proposed an algorithm for a group of agents to track multi-target; Mckenzie (2012) optimised the flocking algorithm in Olfati-Saber (2006) for the case when several flocks meet each other and need to swap their positions; Fan and Zhang (2013) investigated a protocol to solve the problem of a group splitting into two groups flocking in opposite directions. Nevertheless, to the best of our knowledge, few results have discussed the method for a group of agents to disperse and flock in any specified number of groups by self-organising.

Motivated by all the above analysis, and based on a well-known viewpoint of animal behaviour scientists that "schools need no leaders" (Shaw, 1975), we present some distributed algorithms for a group of agents described in double-integrator dynamics to disperse and form several subgroups flocking without leaders asymptotically. In the case of a single group, agents align their velocity vectors by interacting with each other according to simple local rules, and flocking algorithms have been designed to avoid

*Corresponding author. Email: longwang@pku.edu.cn

collisions between any agents and maintain the cohesion of the group (Zavlanos et al., 2007). However, for the case of mixed groups, each agent is affected not only by agents in the same subgroup but also by ones belonging to other subgroups before different subgroups are separated. Furthermore, the algorithms require preserving cohesion of each subgroup while collision avoidance between any two agents in the whole group. These changes induce several difficulties and challenges.

In this paper, the information exchange is undirected in both the same subgroup and the different subgroups. To ensure the high efficiency of information transmission and the reduction of costs in real applications, an algorithm solving the flocking problem for multiple groups of mobile agents in heterogeneous networks is studied. That is, the interactions of velocity and position information employ independent networks (Goldin & Raisch, 2013), where the spread of position information obeys the nearest-neighbour interaction rules. Furthermore, since the communication topology between agents usually changes dynamically in reality, the case of switching velocity topology in heterogeneous networks is also discussed. Finally, an algorithm solving the flocking problem in homogeneous networks is proposed as well. For each given algorithm, the corresponding criteria for flocking are established based on graph theories, matrix theories, and Barbalat's lemma (Khalil, 2002). To deal with the difficulty of discontinuity induced by time-varying communication topology, we design a proper artificial potential function such that the evolution of the states can be studied by Barbalat's lemma.

The rest of the paper is organised as follows. Some basic definitions of graphs and the problem formulation are depicted in Section 2. The main results of flocking in multiple groups are presented in Section 3. Illustrating examples are given in Section 4.

Throughout this paper, let $\mathbb{R}$ be the set of real numbers, and $\mathbb{R}^k$ be the $k-$dimensional Euclidean space, respectively; $||\cdot||$ means the Euclidean norm; $A\backslash B$ denotes the set of those elements of A not belonging to B; X^T stands for the transpose of matrix X; $\mathbf{0}$ represents any zero matrix with an appropriate dimension; $\otimes$ is the kronecker product; $|\mathcal{E}|$ denotes the number of elements in set $\mathcal{E}$; $\forall$ implies for all.

2. Preliminaries and problem formulation

2.1 *Preliminaries of graph theory*

The communications between agents are modelled by a weighted undirected graph. A weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is composed of a set of vertices $\mathcal{V} = \{1, \ldots, n\} \subseteq \mathbb{N}$, a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$. An edge of $\mathcal{G}$ is denoted by (i, j) and a_{ij} denotes the weight of (i, j). For an undirected graph, $(i, j) \in \mathcal{E}$ if and only if $a_{ij} = a_{ji} \neq 0$, which implies agents i and j can interact with each other. Moreover, we assume $a_{ii} = 0$ for any $i \in \mathcal{V}$. Specifically, when we use $(\mathcal{V}, \mathcal{E})$ to denote an undirected graph $\mathcal{G}$, a_{ij} is considered to be 1 if $(i, j) \in \mathcal{E}$. The degree matrix $\Delta = [\Delta_{ij}]$ is a diagonal matrix with $\Delta_{ii} = \sum_{j\in\mathcal{V}} a_{ij}$, and the Laplacian matrix of graph $\mathcal{G}$ is defined by $L = \Delta - \mathcal{A}$. It is obvious that L and $\mathcal{A}$ are both symmetric if $\mathcal{G}$ is undirected; thus, the eigenvalues of L are all real numbers. A path connecting i and j in an undirected graph $\mathcal{G}$ is a sequence of distinct edges of the form $(i_1, i_2), (i_2, i_3), \ldots, (i_{i-1}, i_k)$, where $i_1 = i$, $i_k = j$, and $(i_r, i_{r+1}) \in \mathcal{E}$, $r \in \{1, \ldots, k-1\}$. A graph is said to be connected if there exists a path between any two distinct vertices of the graph. An orientation is the assignment of an arbitrary direction to each edge. The incidence matrix $D = [d_{ij}]$ of an oriented graph is a matrix with rows and columns indexed by the vertices and edges of $\mathcal{G}$, respectively, such that $d_{ij} = 1$ if the vertex i is the head of the edge j, $d_{ij} = -1$ if the vertex i is the tail of the edge j, and $d_{ij} = 0$ otherwise.

In the following, the undirected graphs $\mathcal{G}_v(t) = (\mathcal{V}, \mathcal{E}_v(t), \mathcal{A}(t))$ and $\mathcal{G}_p(t) = (\mathcal{V}, \mathcal{E}_p(t))$ are employed to denote the velocity and position communication topologies, respectively. Let R be the sensing radius of each agent. That is, only agents within distances smaller than R can interact with each other, i.e., $\mathcal{E}_p(t) = \{(i, j)|||x_i(t) - x_j(t)|| < R\}$.

2.2 *Problem formulation*

Consider a multi-agent system with velocity graph $\mathcal{G}_v(t) = (\mathcal{V}, \mathcal{E}_v(t), \mathcal{A}(t))$ and position graph $\mathcal{G}_p(t) = (\mathcal{V}, \mathcal{E}_p(t))$ consisting of s groups of mobile agents moving in a k-dimensional Euclidean space, each agent is described by a double integrator dynamics:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= u_i(t), \quad i \in \mathcal{V}, \end{aligned} \tag{1}$$

where $x_i(t)$, $v_i(t) \in \mathbb{R}^k$ denote the position and velocity of agent i at time t respectively, $u_i(t) \in \mathbb{R}^k$ is the control input for agent i at time t. We now give a definition of flocking in multi-group.

Definition 2.1 (Flocking in multi-group)**:** The problem of flocking in multi-group is solved if a group of mobile agents can be divided into s cohesive subgroups and agents belonging to the same subgroup approach a common velocity vector asymptotically with no collisions occurring in the process, where s is a positive integer.

Suppose the lth subgroup consists of n_l agents, and employs $\mathcal{G}_{pl}(t) = (\mathcal{V}_l, \mathcal{E}_{pl}(t))$ as its position transmission graph, $\mathcal{G}_{vl}(t) = (\mathcal{V}_l, \mathcal{E}_{vl}(t), \mathcal{A}_l(t))$ as its velocity transmission graph, $l \in \{1,\ldots, s\}$. It is straightforward that $\cup_{l=1}^s \mathcal{V}_l = \mathcal{V}$, $\cup_{l=1}^s \mathcal{E}_{pl} \subseteq \mathcal{E}_p$, and $\cup_{l=1}^s \mathcal{E}_{vl} \subseteq \mathcal{E}_v$.

As with the traditional flocking form in Olfati-Saber (2006), Tanner et al. (2007), and Zavlanos et al. (2007), each agent uses a control input consisting of two terms

$$u_i = \alpha_i + \beta_i, \tag{2}$$

where α_i denotes a force directing agent i to align its velocity with other agents in the same subgroup, β_i is a force for collision avoidance and each subgroup's cohesion.

For designing α_i, we use a common velocity consensus term, i.e., $\alpha_i = \sum_{j\in\mathcal{V}} a_{ij}(t)(v_j - v_i)$. In this paper, we hope that each agent aligns its velocity vector with agents in the same subgroup, while neglects the information from other subgroups. Thus, the following assumption for the elements of adjacency matrix $\mathcal{A}(t)$ is made:

$$(A1)\ \sum_{j\in\mathcal{V}\backslash\mathcal{V}_l} a_{ij}(t) = 0, \forall i \in \mathcal{V}_l, l \in \{1, \ldots, s\},$$

which means the total information that each agent in a subgroup achieves from agents in other subgroups is zero. Note that $a_{ij}(t)$ in the assumption is the communication weight in transmission of velocity at time t, it can be negative when agents i and j belong to different subgroups.

For the component β_i, according to Olfati-Saber (2006) and Zavlanos et al. (2007), it is designed as a vector in the direction of the negated gradient of an artificial potential function, which can avoid collisions between agents and maintain links in the position network. For each subgroup, the time-varying set of edges $\mathcal{E}_{pl}(t), l \in \{1, \ldots, s\}$ is further defined as

- $\mathcal{E}_{pl}(0) = \{(i, j) | \|x_i(t) - x_j(t)\| < R, i, j \in \mathcal{V}_l\}$,
- if $0 < \|x_i(t) - x_j(t)\| < r$, then $(i, j) \in \mathcal{E}_{pl}(t)$,
- if $\|x_i(t) - x_j(t)\| \geq R$, then $(i, j) \notin \mathcal{E}_{pl}(t)$,

where $r \in (0, R)$ is a constant value. Such a definition is to induce a hysteresis which is essential for preserving the connectivity of the position graph for each subgroup (Zavlanos et al., 2007). The rest of links in graph $\mathcal{G}_p$ remain the previous definition.

We require collision avoidance between any agents, but not necessarily preserve the connectivity between agents in different subgroups. Based on this consideration, the potential function is designed as follows:

$$V_{ij}(\|x_{ij}\|) = \begin{cases} \dfrac{1}{\|x_{ij}\|^2} + \dfrac{1}{R^2 - \|x_{ij}\|^2}, & (i, j) \in \mathcal{E}_p, i, j \in \mathcal{V}_l, \\ \dfrac{1}{\|x_{ij}\|^2} + \dfrac{2}{R^3}\|x_{ij}\| - \dfrac{3}{R^2}, & (i, j) \in \mathcal{E}_p, i \in \mathcal{V}_l, j \in \mathcal{V}\backslash\mathcal{V}_l, \\ 0, & (i, j) \notin \mathcal{E}_p. \end{cases} \tag{3}$$

where $x_{ij} = x_i - x_j$, $l \in \{1, \ldots, s\}$.

Let $V_i = \sum_{j\in\mathcal{V}, j\neq i} V_{ij}$ be the potential of agent i, and $\beta_i = -\nabla_{x_i} V_i$. Then, we get the control law

$$u_i = \sum_{j\in\mathcal{V}} a_{ij}(t)(v_j - v_i) - \nabla_{x_i} V_i, \quad i \in \mathcal{V}. \tag{4}$$

Using assumptions $(A1)$, the protocol can be rewritten as follows:

$$u_i = \sum_{j\in\mathcal{V}_l} a_{ij}(t)(v_j - v_i) + \sum_{j\in\mathcal{V}\backslash\mathcal{V}_l} a_{ij}(t)v_j - \nabla_{x_i} V_i, \quad i \in \mathcal{V}_l, \quad l \in \{1, \ldots, s\}, \tag{5}$$

V_{ij} defined in Equation (3) allows collision avoidance between any two agents and maintains links in each subgroup. If agents i and j belong to the same subgroup, there are two transition points, *i.e.*, $\|x_{ij}\| = r$ and $\|x_{ij}\| = R$. In this case, for any two agents i and j, $\|x_{ij}\|$ can be guaranteed not to vanish, and it is notable that when $\|x_{ij}\| \to R$, only the loss of (i, j) leads to $V_{ij} \to \infty$, which is the significance of the hysteresis introduced in the definition of $\mathcal{E}_{pl}$, $l \in \{1, \ldots, s\}$. Otherwise, if agents i and j belong to different subgroups, $\|x_{ij}\| = R$ is the only transition point, and the potential function is defined to prevent $\|x_{ij}\|$ going to zero, but allows both the addition and loss of (i, j). Moreover, V_{ij} is set to be differentiable at the point $\|x_{ij}\| = R$, which is crucial for the continuity of the right-hand side of Equation (4).

We aim to separate the s mixed subgroups. Note that for agents in different subgroups, the artificial potential function V_{ij} produces a repulsive force when $\|x_{ij}\| < R$, and vanishes when $\|x_{ij}\| \geq R$, for i and j belonging to different subgroups.

Lemma 2.2 (Barbalat's lemma, Khalil, 2002)**:** *Let $V(t)$ be a differentiable function and it has a finite limit as $t \to \infty$, if $\dot{V}(t)$ is uniformly continuous with respect to t, then* $\lim_{t\to\infty} \dot{V}(t) = 0$.

3. Main results

3.1 *Flocking in heterogeneous networks*

In this subsection, we consider the mobile agents interacting in heterogeneous networks with fixed and switching velocity topology, and assume the initial position graph of each subgroup $\mathcal{G}_{pl}(0)$, $l \in \{1, \ldots, s\}$ is connected. Before showing main theorems, we propose the following proposition:

Proposition 3.1: *For system* (1) *with protocol* (4)*, assume that the Laplacian matrix $L(t)$ of velocity graph $\mathcal{G}_v$ has exactly s simple zero eigenvalues and all the other eigenvalues are positive real numbers for all $t \geq 0$. Then, the edges of $\mathcal{G}_{pl}(t)$, $l \in \{1, \ldots, s\}$ will never vanish at all times.*

Proof: Let $\bar{\mathbf{x}}$ denote a vector which consists of the position difference between any two agents in the whole group, i.e., $\bar{\mathbf{x}} = (D^T \otimes I_k)\mathbf{x}$, where $\mathbf{x}$ is a position vector composed of

$\sum_{l=1}^{s} n_l$ components, D is the oriented incidence matrix of the complete graph with vertices set $\mathcal{V}$. Then, system (1) with protocol (5) is transformed to the following system:

$$\dot{\bar{\mathbf{x}}} = (D^T \otimes I_k)\mathbf{v}, \\ \dot{\mathbf{v}} = \mathbf{u}, \tag{6}$$

where $\mathbf{v}$ denotes the velocity vector, $\mathbf{u}$ is the control input vector. For system (6), we construct an energy function W as follows:

$$W(\bar{\mathbf{x}}, \mathbf{v}) = \frac{1}{2}||\mathbf{v}||^2 + \frac{1}{2}\sum_{i\in\mathcal{V}} V_i. \tag{7}$$

Let t_i, $i = 1, 2, \ldots$ be the time at which the velocity graph or the position graph switches, and $t_0 = 0$ be the initial time. For any $t \in [t_i, t_{i+1})$, $i = 0, 1, 2, \ldots$, it is straightforward that W is differentiable, $a_{ij}(t)$ and $L(t)$ are constant, implying that

$$\dot{W} = \sum_{i\in\mathcal{V}} v_i^T \left(\sum_{j\in\mathcal{V}} a_{ij}(t)(v_j - v_i) - \nabla_{x_i} V_i \right) + \frac{1}{2}\sum_{i\in\mathcal{V}} \dot{V}_i,$$

From the definition of V_{ij}, one has

$$\begin{aligned} \frac{1}{2}\sum_{i\in\mathcal{V}} \dot{V}_i &= \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}} \dot{x}_{ij}^T \nabla_{x_{ij}} V_{ij} \\ &= \frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}} \left(\dot{x}_i^T - \dot{x}_j^T\right)\nabla_{x_{ij}} V_{ij} \\ &= \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}} v_i^T \nabla_{x_i} V_{ij} = \sum_{i\in\mathcal{V}} v_i^T \nabla_{x_i} V_i. \end{aligned} \tag{8}$$

Then

$$\begin{aligned} \dot{W} &= \sum_{i\in\mathcal{V}} v_i^T \left(\sum_{j\in\mathcal{V}} a_{ij}(t)(v_j - v_i) - \nabla_{x_i} V_i \right) + \sum_{i\in\mathcal{V}} v_i^T \nabla_{x_i} V_i \\ &= \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}} a_{ij}(t) v_i^T (v_j - v_i) \\ &= -\mathbf{v}^T (L(t) \otimes I_k)\mathbf{v}, \end{aligned} \tag{9}$$

where $L(t)$ is the Laplacian matrix of velocity graph $\mathcal{G}_v$. When $L(t)$ has exactly s simple zero eigenvalues and the rest are positive real numbers, it is clear that $L(t)$ is a positive semi-definite matrix. Then $\dot{W} \le 0$ for $t \in [t_i, t_{i+1})$, $i = 0, 1, 2, \ldots$, which induces $\dot{W} \le 0$ for all $t \ge 0$. Consequently, there exists a positive constant value c, such that $V_{ij} \le c$, $i, j \in \mathcal{V}_A$ at all times. For any $l \in \{1, \ldots, s\}$, assume that a link is lost in graph $\mathcal{G}_{pl}$ at $t = t'$, which means that $\|x_{ij}(t')\| = R$, and $(i, j) \in \mathcal{E}_{pl}$ at the previous moment. It follows that $V_{ij}(t) \to \infty$ as $t \to t'$, which is a contradiction. Thus, all the links in graph $\mathcal{G}_{pl}$ can be maintained. □

Remark 1: Note that Proposition 3.1 is applicable to the case of heterogeneous networks with both fixed and switching velocity topology, as well as the case of homogeneous networks since the proposition allows velocity and position topologies to switch independently.

For the case that the velocity graph is time-invariant, i.e., $\mathcal{G}_v(t) = \mathcal{G}_v = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, the following theorem is presented:

Theorem 3.2: *Consider a group of mobile agents* (1) *with protocol* (4). *Suppose* ($A1$) *holds and the initial position graph* $\mathcal{G}_{pl}(0)$ *is connected for any* $l \in \{1, \ldots, s\}$. *If the Laplacian matrix L of velocity graph* $\mathcal{G}_v$ *has exactly s simple zero eigenvalues and all the other eigenvalues are positive real numbers, then flocking in multi-group can be solved.*

Proof: Our first goal is to show that agents in each subgroup attain a common velocity vector asymptotically.

For any $l \in \{1, \ldots, s\}$, by Proposition 3.1, together with the connectivity of the initial position graph, one obtains the fact that $|\mathcal{E}_{pl}(t)|$ is nondecreasing, and $\mathcal{G}_{pl}(t)$ always preserves its connectivity. Note that $|\mathcal{E}_{pl}(t)| \le \frac{n_l(n_l-1)}{2}$, then $|\mathcal{E}_{pl}(t)|$ converges to constant values, implying that $\mathcal{G}_{pl}(t)$ switches finite times. We set its final topology as $\mathcal{G}_{pl}(t^*)$. For the links between agents from different subgroups, V_{ij} is differentiable at the transition point ($\|x_{ij}\| = R$), and thus does not introduce discontinuities when the position topology switches. That is, when we study the problem on the interval (t^*, ∞), the right-hand side of Equation (4) is continuous, which implies that for all $i \in \mathcal{V}$, v_i is continuous and $W(t)$ is differentiable.

According to Equations (7) and (9), we have $W \ge 0$ and $\dot{W} \le 0$ at all times, then W has a finite limit as $t \to \infty$. To employ Lemma 2.2, we need the uniform continuity of $\dot{W}$. It follows from Equation (9) that

$$\dot{W} = -\mathbf{v}^T (L \otimes I_k)\mathbf{v} = -\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}} a_{ij}||v_i - v_j||^2.$$

For any $0 \le t_1 < t_2$, we have

$$\begin{aligned} |\dot{W}(t_1) - \dot{W}(t_2)| &= \Bigg| -\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}} a_{ij}||v_i(t_1) - v_j(t_1)||^2 \\ &\quad + \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}} a_{ij}||v_i(t_2) - v_j(t_2)||^2 \Bigg| \\ &= \Bigg| \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}} a_{ij}\big(||v_i(t_1) - v_j(t_1)|| + ||v_i(t_2) - v_j(t_2)||\big) \cdot \\ &\quad \big(||v_i(t_1) - v_j(t_1)|| - ||v_i(t_2) - v_j(t_2)||\big) \Bigg|, \end{aligned} \tag{10}$$

where

$$\big|\, ||v_i(t_1) - v_j(t)|| - ||v_i(t_2) - v_j(t_2)|| \,\big| \\ \leq ||v_i(t_1) - v_j(t) - v_i(t_2) + v_j(t_2)||.$$

We now proceed to prove v_i is uniformly continuous with respect to t for any $i \in \mathcal{V}$.

From Equation (9), there exists a positive constant value c, such that $||\mathbf{v}|| \leq c$ and $\sum_{i \in \mathcal{V}} V_i \leq c$, which leads to $||v_i|| \leq c$, $V_{ij} \leq c, \forall i, j \in \mathcal{V}$. Note that

$$||\nabla_{x_i} V_{ij}|| = ||\nabla_{x_{ij}} V_{ij}|| = \left|\left| \frac{dV_{ij}}{d||x_{ij}||} \cdot \nabla_{x_{ij}} ||x_{ij}|| \right|\right| \\ = \left| \frac{dV_{ij}}{d||x_{ij}||} \right|.$$

We use this fact and the definition of V_{ij} to obtain the boundedness of $||\nabla_{x_i} V_{ij}||$. As a result,

$$||\dot{v}_i|| = \left|\left| \sum_{j \in \mathcal{V}} a_{ij}(v_j - v_i) - \sum_{j \in \mathcal{V}, j \neq i} \nabla_{x_i} V_{ij} \right|\right| \\ \leq \sum_{j \in \mathcal{V}, j \neq i} |a_{ij}|(||v_i|| + ||v_j||) + \sum_{j \in \mathcal{V}, j \neq i} M \\ \leq \left(\sum_{l=1}^{s} n_l - 1 \right)(2ac + M), \tag{11}$$

where M is the upper bound of $||\nabla_{x_i} V_{ij}||$, $a = \max_{i,j \in \mathcal{V}}\{|a_{ij}|\}$. Together with the continuity of v_i, the uniform continuity of v_i for $t > t^*$ now follows.

Consequently, for any $t_1, t_2 > 0$, $i \in \mathcal{V}$, one has $\lim_{|t_1 - t_2| \to 0} ||v_i(t_1) - v_i(t_2)|| = 0$, it follows from Equation (10) that $\lim_{|t_1 - t_2| \to 0} \left| \dot{W}(t_1) - \dot{W}(t_2) \right| = 0$. This yields the uniform continuity of $\dot{W}$ for $t > t^*$. By Lemma 2.2, we obtain $\lim_{t \to \infty} \dot{W} = \lim_{t \to \infty} \mathbf{v}^T (L \otimes I_k)\mathbf{v} = 0$. On the other hand, when assumption ($A1$) is satisfied, the Laplacian matrix L has at least s linearly independent eigenvectors, *i.e.*, $(\underbrace{1, \ldots, 1}_{n_1}, \underbrace{0, \ldots, 0}_{n_2 + \cdots + n_s})^T, \ldots, (\underbrace{0, \ldots, 0}_{n_1 + \cdots + n_{s-1}}, \underbrace{1, \ldots, 1}_{n_s})^T$ associated with the zero eigenvalue. Owing to the condition that L has exactly s simple zero eigenvalues and the rest are positive real numbers, L has exactly these s eigenvectors associated with the zero eigenvalue. Hence, $\lim_{t \to \infty} ||v_i(t) - v_j(t)|| = 0$, $i, j \in \mathcal{V}_l$, $l \in \{1, \ldots, s\}$.

The cohesion of each subgroup is obvious since the position graph of each subgroup is connected at all times.

Finally, we have to show collision avoidance in the whole group. For any $(i, j) \in \mathcal{E}_p$, if $\|x_i - x_j\| \to 0$, one has $V_{ij} \to \infty$, which contradicts with $V_{ij} \leq c$. Therefore, no collisions occur between any agents.

Remark 2: For $t > t^*$, the continuity of $\frac{dV_{ij}}{d||x_{ij}||}$ at $\|x_{ij}\| = R$ is essential to the continuity of $\dot{v}_i$, which leads to the continuity of v_i, and it follows the uniform continuity of v_i. According to the fact that the existence of a continuous solution is not always guaranteed for a discontinuous dynamical system Cortes (2008), for an easier analysis, we study the problem for $t > t^*$ so that the system is continuous and the position topology becomes fixed. This implies that a continuously differentiable solution starts at t^* always exists, and Barbalat's lemma can be used.

When the velocity graph is time-varying, *i.e.*, $\mathcal{G}_v(t) = (\mathcal{V}, \mathcal{E}_v(t), \mathcal{A}(t))$. We make the following assumption further:

($A2$) Let t_i for $i = 1, 2, \ldots$ denote the time when the topology of $\mathcal{G}_v(t)$ changes, t_0 is the initial time, there exists a $\tau > 0$ such that $t_{i+1} - t_i \geq \tau$, $\forall i = 0, 1, 2, \ldots$.

Then we present the result as follows:

Theorem 3.3: *Consider a group of mobile agents* (1) *with protocol* (4). *Suppose* ($A1 - A2$) *hold and the initial position graph* $\mathcal{G}_{pl}(0)$ *is connected for any* $l \in \{1, \ldots, s\}$. *If the Laplacian matrix* $L(t)$ *of velocity graph* $\mathcal{G}_v(t)$ *has exactly s simple zero eigenvalues and all the other eigenvalues are positive real numbers for all* $t \geq 0$, *then flocking in multi-group can be solved.*

Proof: As analysed in the proof of Theorem 3.2, we study this problem when $t > t^*$, and it is easy to get that $\dot{W}$ is uniformly continuous with respect to t for $t \in [t_i, t_{i+1})$, $t_i > t^*$. Now let us prove $\dot{W} \to 0$, as $t \to \infty$. Suppose this is not true. Then, there exists a constant $\varepsilon > 0$ such that $\dot{W}(b_i) < -\varepsilon$, $i = 1, 2, \ldots$, where $\{b_1, b_2, \ldots\}$ is an infinite sequence and $b_i \in [t_{m_i}, t_{m_i+1})$, $t_{m_i} > t^*$. From the uniform continuity of $\dot{W}$ in $[t_{m_i}, t_{m_i+1})$, there exists a constant $0 < \delta < \tau$ such that each b_i is contained in an interval of length δ in which $\dot{W}(t) \leq -\varepsilon < -\varepsilon/2$. Let $M > \frac{2W(t^*)}{\delta\varepsilon}$, for $t > t_{m_M+1}$, we have

$$W(t) = W(t^*) + \int_{t^*}^{t} \dot{W}(s)ds \\ < W(t^*) - \frac{M\delta\varepsilon}{2} < 0, \tag{12}$$

which conflicts with the fact that W is positive semi-definite of t.

Hence, we obtain $\lim_{t \to \infty} \dot{W}(t) = 0$, *i.e.*, $\lim_{t \to \infty} \mathbf{v}^T (L(t) \otimes I_k)\mathbf{v} = 0$. Due to the property of $L(t)$, it has exactly s linearly independent eigenvectors, *i.e.*, $(\underbrace{1, \ldots, 1}_{n_1}, \underbrace{0, \ldots, 0}_{n_2 + \cdots + n_s})^T, \ldots, (\underbrace{0, \ldots, 0}_{n_1 + \cdots + n_{s-1}}, \underbrace{1, \ldots, 1}_{n_s})^T$ associated with the s simple zero eigenvalues. Thus, one has $\lim_{t \to \infty} ||v_i(t) - v_j(t)|| = 0$, $i, j \in \mathcal{V}_l$, $l \in \{1, \ldots, s\}$. The remainder of the argument is analogous to that in Theorem 3.2 and is omitted here. □

Remark 3: In the above theorems, the assumption for eigenvalues of $L(t)$ can be transformed to restrictions on graphs in a special case. In fact, for any fixed time t, when each agent receives no velocity information from agents belonging to other subgroups, *i.e.*, $a_{ij}(t)=0, i \in \mathcal{V}_l, j \in \mathcal{V}\backslash\mathcal{V}_l$, $l \in \{1, \cdots, s\}$, implying that $(A1)$ is distinctly satisfied and velocity graph $\mathcal{G}_v$ has at least s connected components, if $\mathcal{G}_{vl}(t)$ is connected for all $l \in \{1,\ldots,s\}$, then $L(t)$ has exactly s zero eigenvalues and all the other eigenvalues are positive real numbers. This is due to the fact that $L(t)$ becomes a block s diagonal matrix, where each submatrix has exactly one zero eigenvalue and the rest of eigenvalues are positive real numbers.

To derive criteria associated with graphs for all cases, we modify protocol (4) to make each agent neglect the velocity information from agents in other subgroups. The protocol is replaced by the following form:

$$u_i = \sum_{j\in\mathcal{V}_l} a_{ij}(t)(v_j - v_i) - \nabla_{x_i} V_i, \;\; i \in \mathcal{V}_l, \;\; l \in \{1, \ldots, s\}, \tag{13}$$

where $V_i = \sum_{j\in\mathcal{V}, j\neq i} V_{ij}$, V_{ij} is defined in Equation (3). When protocol (13) is applied, assumption $(A1)$ is no longer required since $a_{ij}(t)$ with $i \in \mathcal{V}_l$, $j \in \mathcal{V}\backslash\mathcal{V}_l$ has no affect on the system. Together with the analysis in Remark **3**, we have the following result.

Corollary 3.4: *Consider a group of mobile agents* (1) *with protocol* (13). *Suppose* $(A2)$ *holds, the initial position graph* $\mathcal{G}_{pl}(0)$ *and velocity graph* $\mathcal{G}_{vl}(t)$ *are both connected for any* $l \in \{1, \ldots, s\}$, $t \geq 0$, *then flocking in multi-group can be solved.*

3.2 *Flocking in homogeneous networks*

In this subsection, the information transmission of velocity and position is under the same communication topology, i.e., $\mathcal{G}_v(t) = \mathcal{G}_p(t) = \mathcal{G}(t) = (\mathcal{V}, \mathcal{E}_p(t))$. To separate the whole group, we make each agent not use the velocity information from other subgroups. The protocol is proposed as follows:

$$u_i = \sum_{j\in\mathcal{N}_i(t)\cap\mathcal{V}_l} a_{ij}(t)(v_j - v_i) - \nabla_{x_i} V_i, \;\; i \in \mathcal{V}_l, \quad l \in \{1, \ldots, s\}, \tag{14}$$

where $V_i = \sum_{j\in\mathcal{V}, j\neq i} V_{ij}$, V_{ij} is defined in Equation (3), $\mathcal{N}_i(t) = \{j|(i, j) \in \mathcal{E}_p(t), j \neq i, j \in \mathcal{V}\}$ denotes the set of neighbours of agent i at time t. Our result is stated in the following theorem:

Theorem 3.5: *Consider a group of mobile agents* (1) *with protocol* (14). *Assume the initial position graphs* $\mathcal{G}_{pl}(0)$, $l \in \{1, \ldots, s\}$ *are all connected. Then, flocking in multi-group can be solved.*

Proof: As explained in the proof of Theorem 3.2, there exists a time t^*, such that the communication topology becomes fixed at time t^*, i.e., $\mathcal{G}_l(t) \to \mathcal{G}_l(t^*)$, $l \in \{1, \ldots, s\}$. We now discuss the problem once the topology stops switching. Note that protocol (14) is a special case of Equation (4), and thus Proposition 3.1 is applicable. Then we can obtain

$$\dot{W} = -\mathbf{v}^T (L(t) \otimes I_k)\mathbf{v} \leq 0,$$

where $L(t) = \begin{pmatrix} L_1(t) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & L_s(t) \end{pmatrix}$, $L_l(t)$ is the Laplacian matrix of $\mathcal{G}_l(t)$, $l \in \{1,\ldots, s\}$.

For $t > t^*$, $L_l(t) = L_l(t^*)$, *i.e.*, $L(t) = L(t^*) = \begin{pmatrix} L_1(t^*) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & L_s(t^*) \end{pmatrix}$. As with the method used in the proof of Theorem 3.2, it is easy to show the uniform continuity of $\dot{W}(t)$ with respect to t for $t > t^*$. Then, by Barbalat's lemma, one has $\lim_{t\to\infty} \dot{W}(t) = 0$. That is, $\lim_{t\to\infty} \mathbf{v}^T (L(t^*) \otimes I_k)\mathbf{v} = 0$. For any $l \in \{1,\ldots, s\}$, from the connectivity of $\mathcal{G}_l(t^*)$, $L_l(t^*)$ has a simple zero eigenvalue, and the corresponding eigenvector is $(\underbrace{1, \ldots, 1}_{n_l})^T$. Thus, we obtain $\lim_{t\to\infty} ||v_i(t) - v_j(t)|| = 0$, $i, j \in \mathcal{V}_l$, $l \in \{1, \ldots, s\}$. The remainder of the argument is analogous to that in Theorem 3.2 and is omitted here. □

The result in homogeneous networks imposes no restrictions on velocity graphs, so that it is easier to verify the convergence of flocking by using protocol (14). However, in real applications, multiple information usually cannot be transmitted accurately at once, and efficient homogeneous networks may take more costs than heterogeneous networks do. Thus, both heterogeneous networks and homogeneous networks have their own advantages.

Remark 4: Using protocol (4), a special case cannot be ignored. In fact, when some agents in a subgroup are surrounded by the agents in another subgroup, the two subgroups are possibly inseparable at all times. For instance, the two groups of agents depicted in Figure 1 will never disperse. In this case, the aforementioned theorems also hold. That is, $\lim_{t\to\infty} ||v_i(t) - v_j(t)|| = 0$, $i, j \in \mathcal{V}_l$, $l \in \{A, B\}$. It is straightforward to show that all the agents will achieve a common velocity by reduction to absurdity; therefore, the problem of flocking in a single group is solved simultane-

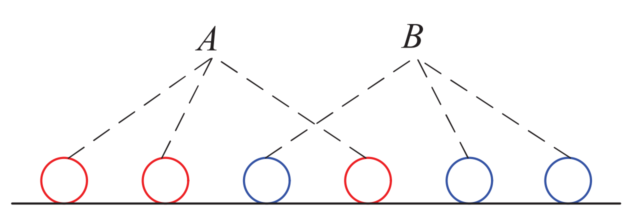

Figure 1. Two groups of agents move on a line.

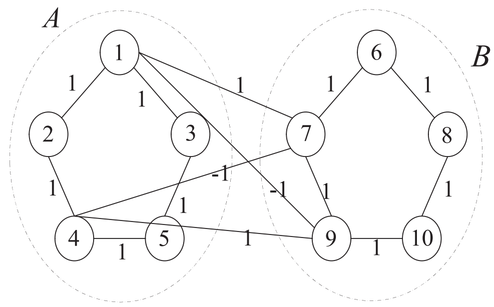

Figure 2. The fixed velocity graph $\mathcal{G}_v$.

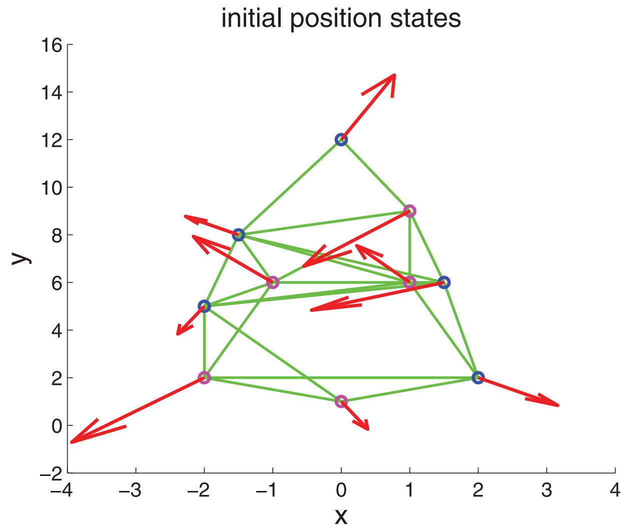

Figure 3. The initial position states of agents.

ously. It is notable that such a case only happens when the velocity vectors of all agents are colinear and parallel to the line connected between any two agents in the group; the probability of this happening in Euclidean space $\mathbb{R}^k$, $k \geq 2$ is rare. Moreover, in real applications, the whole group can be artificially divided to avoid this special case.

Remark 5: When $l \in \{1\}$, algorithms (5) and (14) will reduce to the ones for the traditional flocking problem, which are similar to the algorithms described in Tanner et al. (2007) and Zavlanos et al. (2007), respectively.

4. Simulations

In this section, we present several simulation results to illustrate the effectiveness of our theoretical results. In simulations, we set the sensing radius $R = 5$, and hysteresis $r = 4$. The initial position vector of each agent is restricted to ensure the connectivity of the position graph of each subgroup, the initial velocity vector of each agent is selected randomly.

Example 4.1: We consider a group of 10 agents including two subgroups that each subgroup consists of 5 agents moving in the plane. Figure 2 shows the undirected velocity interaction graph $\mathcal{G}_v$; it is easy to get the corresponding Laplacian matrix L. The numerical computation shows that L has exactly two simple zero eigenvalues and all the other eigenvalues are positive real numbers. Figure 3 depicts the

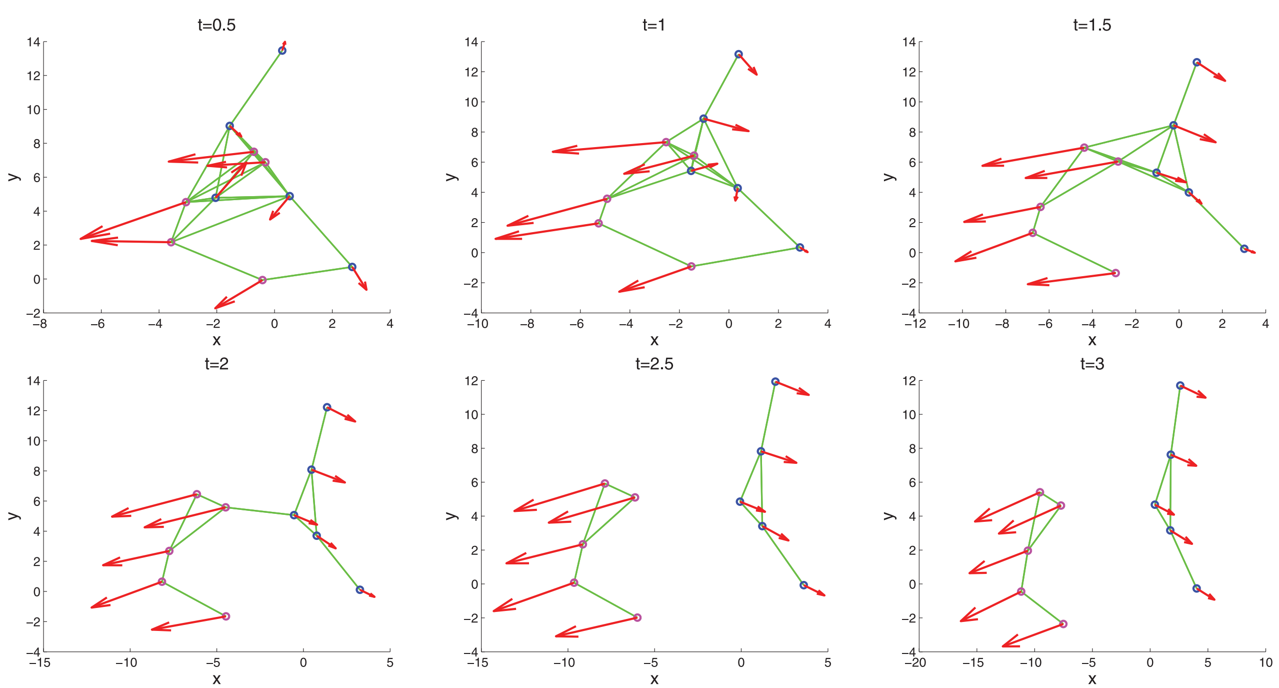

Figure 4. Snapshots of flocking in a couple of groups with fixed velocity topology.

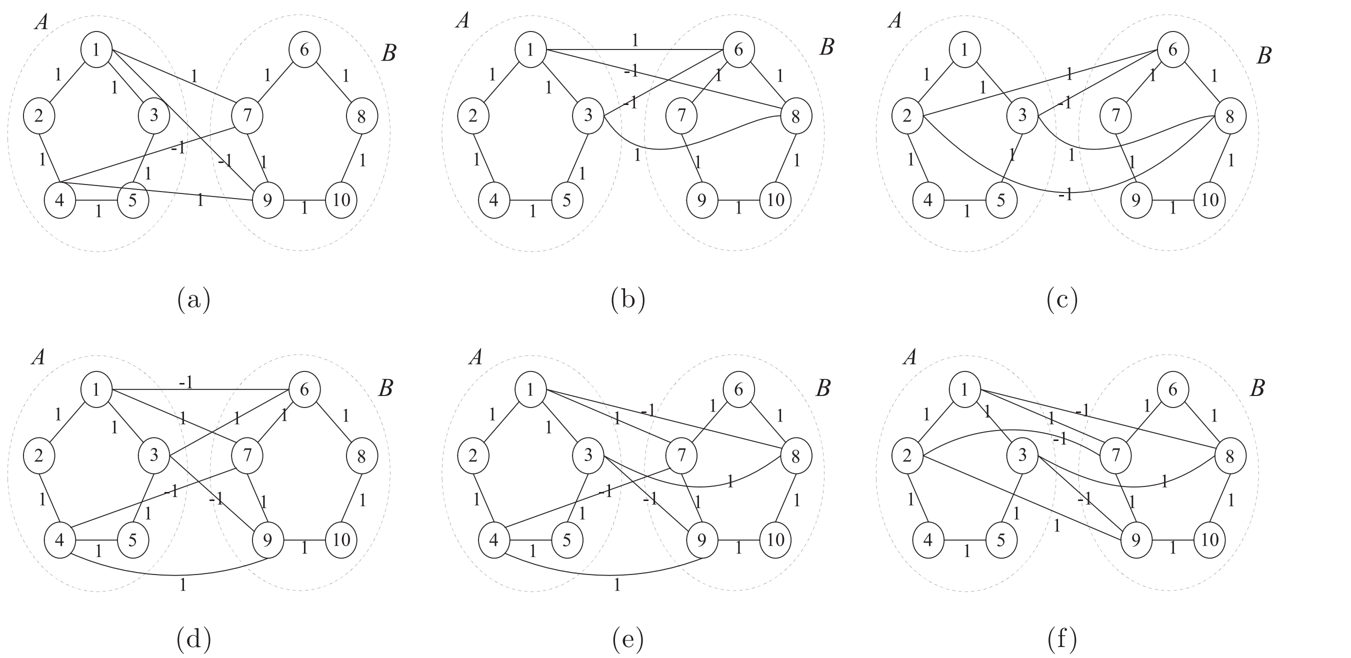

Figure 5. Six undirected connected graphs satisfying ($A1$) and ($A2$).

initial position states of the agents. Figure 4 describes the evolution of agents applying algorithm (4). The colour of each point in Figures 3 and 4 represents the subgroup the agent belongs to, the arrow on each point shows its velocity vector at the corresponding time, and a green line implies connected. It is clear that the flocking problem for two groups of agents is solved, and the connectivity of each subgroup's position graph is maintained at all times, which implies cohesion.

Example 4.2: Now we consider the behaviour of 10 agents in heterogeneous networks with switching velocity interac-

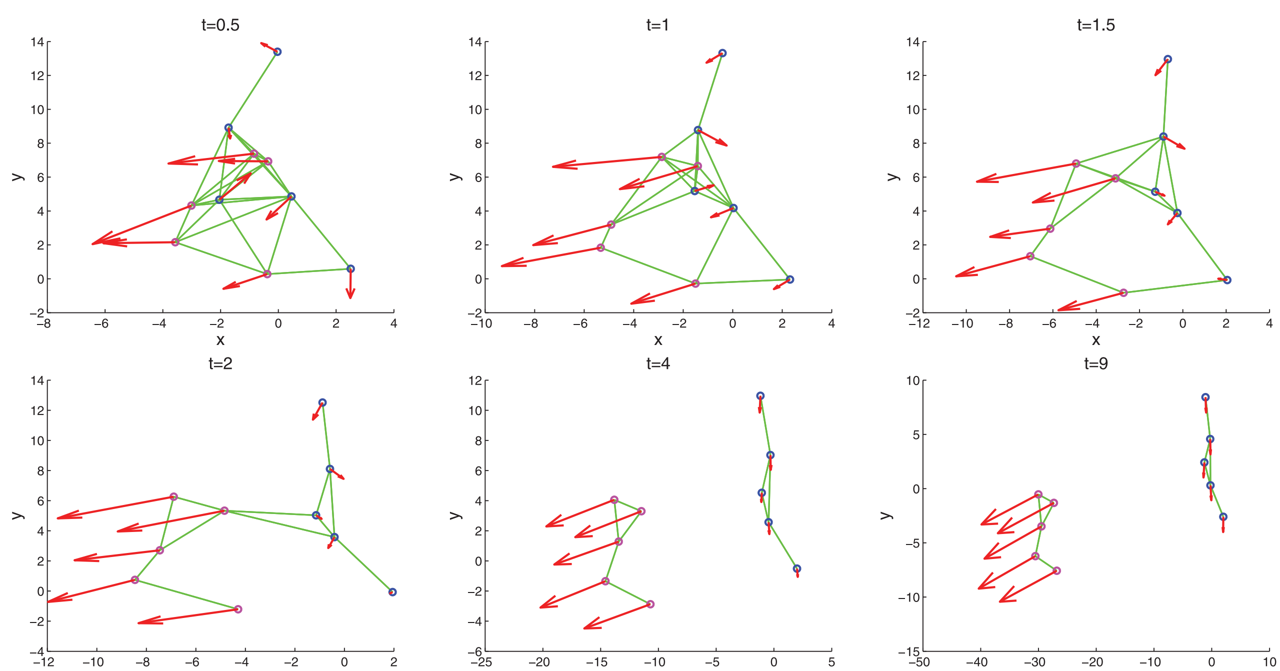

Figure 6. Snapshots of flocking in a couple of groups with switching velocity topology.

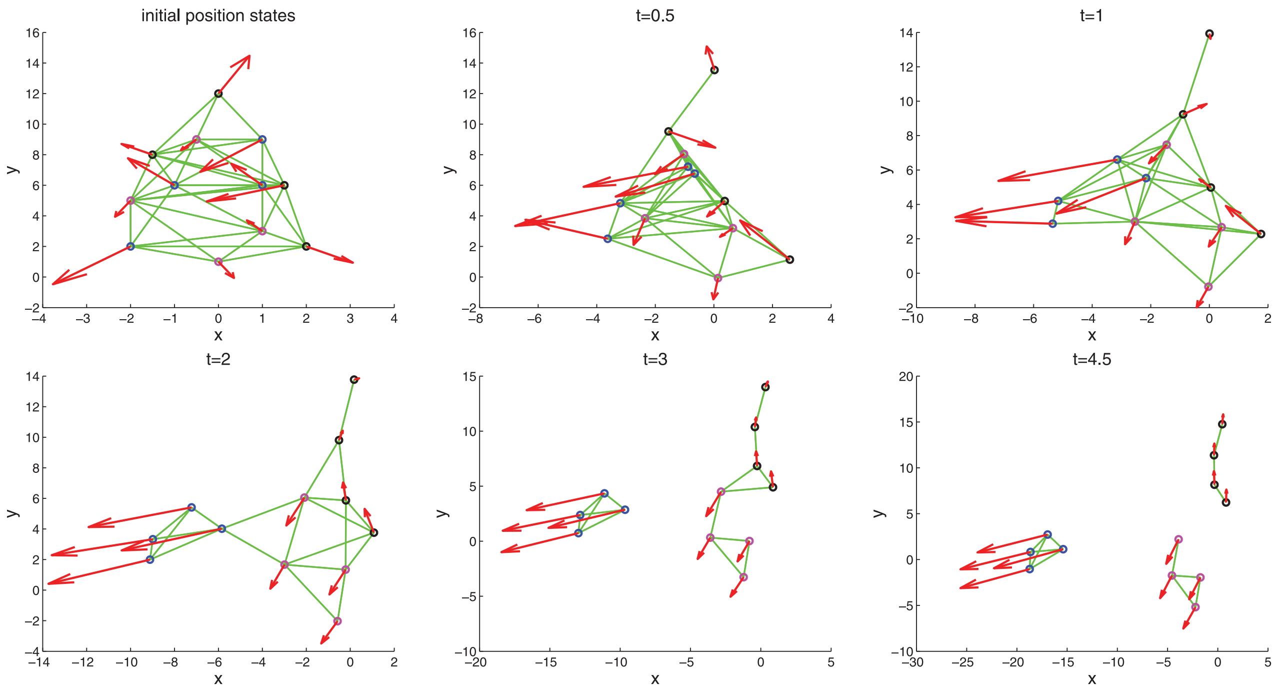

Figure 7. Snapshots of flocking in three groups when the networks are homogeneous.

tion graph. In this case, some of the existing interaction links fail and some new edges are created due to the influence of external factors. We set the dwell time between any two adjacent switching time as 0.005s, the velocity graph is chosen from $\{\mathcal{G}_a, \mathcal{G}_b, \ldots, \mathcal{G}_f\}$ in order (see Figure 5). For each velocity graph, the corresponding Laplacian matrix L has exactly two simple zero eigenvalues and the rest are positive real numbers. The initial position states of the agents are the same as the ones in Example 1. The 10 agents applying protocol (4) successfully flock in two groups of agents and the links between agents in each subgroup are maintained, as shown in Figure 6.

Example 4.3: This example is for illustrating the result of flocking in homogeneous networks. In this case, the group of 12 agents consist of three subgroups. It can be seen from Figure 7 that by applying protocol (14), the flocking problem described in Definition 2.1 is solved.

5. Conclusion

In this paper, the problem of flocking in multiple groups of agents has been solved, which describes that a group of agents disperse to several cohesive subgroups with each subgroup achieving a common velocity gradually, while the collisions between any agents are avoided. The distributed feedback designs have been presented in both heterogeneous networks and homogeneous networks. For each flocking algorithm, we have provided the corresponding conditions under which the flocking problem is solved. Moreover, the theoretical results have been numerically verified.

Since our algorithms are the extensions of those for the traditional flocking problem, a number of works of the latter may be feasible for the former and are worth exploring. Such results include flocking with nonlinear measurements, the asymmetric interaction of velocity in heterogeneous networks, the intermittent control input, and so on.

Acknowledgements

The authors would like to acknowledge Dr Te Wu and Dr Jingying Ma for inspiring discussions on this topic.

Funding

This work was supported by NSFC [grant number 61020106005], [grant number 61375120], [grant number 61304160]; 973 Program [grant number 2012CB821203]; the Fundamental Research Funds for the Central Universities [grant number JB140406].

References

Akyildiz, I.F., Su, W., Sankarasubramaniam, Y., & Cayirci, E. (2002). A survey on sensor networks. *IEEE Communications Magazine, 40*(8), 102–114.

Altafini, C. (2013). Consensus problems on networks with antagonistic interactions. *IEEE Transactions on Automatic Control, 58*(4), 935–946.

Blondel, V.D., Hendrickx, J.M., & Tsitsiklis, J.N. (2009). On Krause's multi-agent consensus model with state-dependent

connectivity. *IEEE Transactions on Automatic Control, 54*(11), 2586–2597.
Cortes, J. (2008). Discontinuous dynamical systems. *IEEE Control Systems, 28*(3), 36–73.
Cucker, F., & Dong, J.G. (2010). Avoiding collisions in flocks. *IEEE Transactions on Automatic Control, 55*(5), 1238–1243.
Cucker, F., & Smale, S. (2007). Emergent behavior in flocks. *IEEE Transactions on Automatic Control, 52*(5), 852–862.
Fan, M.C., & Zhang, H.T. (2013). Bipartite flock control of multi-agent systems. In *The 32nd Chinese Control Conference, 26–28 July,* (pp. 6993–6998). Xi'an.
Goldin, D., & Raisch, J. (2013). Consensus for agents with double integrator dynamics in heterogeneous networks. *Asian Journal of Control, 15*(4), 1–10.
Hegselmann, R., & Krause, U. (2002). Opinion dynamics and bounded confidence models, analysis, and simulation. *Journal of Artificial Societies and Social Simulation*, *5*(3), Retrieved from http://jasss.soc.surrey.ac.uk/9/1/8.html.
Ji, Z., Wang, Z., Lin, H., & Wang, Z. (2010). Controllability of multi-agent systems with time-delay in state and switching topology. *International Journal of Control, 83*(2), 371–386.
Khalil, H.K. (2002). *Nonlinear systems* (Vol. 3). Upper Saddle River: Prentice Hall.
Luo, X., Li, S., & Guan, X. (2010). Flocking algorithm with multi-target tracking for multi-agent systems. *Pattern Recognition Letters, 31*(9), 800–805.
Mckenzie, A.W. (2012). *Robotic control: Real-time architectures and multi-flock flocking* (PhD thesis). The University of Alabama, Tuscaloosa, AL.
Okubo, A. (1986). Dynamical aspects of animal grouping: Swarms, schools, flocks, and herds. *Advances in Biophysics, 22*, 1–94.
Olfati-Saber, R. (2006). Flocking for multi-agent dynamic systems: Algorithms and theory. *IEEE Transactions on Automatic Control, 51*(3), 401–420.
Ren, W., & Beard, R. (2008). *Distributed consensus in multi-vehicle cooperative control: Theory and applications*. London: Springer-Verlag.
Reynolds, C.W. (1987). Flocks, herds, and schools: A distributed behavioral model. *Computer Graphics, 21*(4), 25–34.
Shaw, E. (1975). Fish in schools. *Natural History, 84*(8), 40–45.
Shi, H., Wang, L., & Chu, T. (2006). Virtual leader approach to coordinated control of multiple mobile agents with asymmetric interactions. *Physica D: Nonlinear Phenomena, 213*(1), 51–65.
Su, H., Wang, X., & Yang, W. (2008). Flocking in multi-agent systems with multiple virtual leaders. *Asian Journal of Control, 10*(2), 238–245.
Tanner, H.G., Jadbabaie, A., & Pappas, G.J. (2007). Flocking in fixed and switching networks. *IEEE Transactions on Automatic Control, 52*(5), 863–868.
Vicsek, T., Czirk, A., Ben-Jacob, E., Cohen, I., & Shochet, O. (1995). Novel type of phase transition in a system of self-driven particles. *Physical Review Letters, 75*(6), 1226–1229.
Xiao, F., Wang, L., Chen, J., & Gao, Y. (2009). Finite-time formation control for multi-agent systems. *Automatica, 45*(11), 2605–2611.
Yu, J., & Wang, L. (2009). Group consensus of multi-agent systems with undirected communication graphs. In *The 7th Asian Control Conference, 27–29 August,* (pp. 105–110). Hong Kong.
Yu, J., & Wang, L. (2010). Group consensus in multi-agent systems with switching topologies and communication delays. *Systems and Control Letters, 59*(6), 340–348.
Zavlanos, M.M., Jadbabaie, A., & Pappas, G.J. (2007). Flocking while preserving network connectivity. In *The 46th IEEE Conference on Decision and Control, 12–14 December,* (pp. 2919–2924). New Orleans, LA.
Zheng, Y., & Wang, L. (2012). Distributed consensus of heterogeneous multi-agent systems with fixed and switching topologies. *International Journal of Control, 85*(12), 1967–1976.
Zheng, Y., Zhu, Y., & Wang, L. (2011). Consensus of heterogeneous multi-agent systems. *IET Control Theory and Applications, 5*(16), 1881–1888.