Online Distributed Optimization With Strongly Pseudoconvex-Sum Cost Functions

Kaihong Lu, Gangshan Jing, and Long Wang

***Abstract*—In this paper, the problem of online distributed optimization is investigated, where the sum of locally dynamic cost functions is considered to be strongly pseudoconvex. To address this problem, we propose an online distributed algorithm based on an auxiliary optimization strategy. The algorithm involves each agent minimizing its own cost function subject to a common convex set while exchanging local information with others under a time-varying directed communication graph sequence. The dynamic regret is employed to measure performance of the algorithm. Under mild conditions on the graph, it is shown that if the increasing rate of minimizer sequence's deviation is within a certain range, then the bound of each dynamic regret function grows sublinearly. Simulations are presented to demonstrate the effectiveness of our theoretical results.**

***Index Terms*—Consensus, dynamic regret, multiagent systems, online distributed optimization.**

I. Introduction

Along with the penetration of multiagent networks [1], [7], distributed optimization via a network of agents has received increasing attention in recent years [2]–[6], where the goal of agents is to minimize the global cost function formed by the sum of local functions via local information. This is due to its wide practical applications including distributed localization and estimation [8], energy dispatch in power distribution networks [9], and distributed machine learning [10].

Recently, some results on online distributed optimization have been achieved, where cost functions vary over time and changes can only be seen by agents in hindsight. It is necessary for an online algorithm to mimic the performance of its offline counterpart, and the gap between them is called the regret. In [11]–[15], the performance of an online algorithm is studied by a static regret, where the offline problem is to minimize the sum of global cost functions at all time. Under the presented algorithms in [11]–[15], the bound of the static regret just increases sublinearly.

Dynamic environments arise in many practical applications. For example, when tracking a moving target, one should solve an online optimization problem where both loss functions and comparators are time-varying [16]. To adapt to such dynamic environments, dynamic regrets, where the benchmark is to minimize the global cost function at each time, are required to measure the performance of online optimization algorithms [17]–[20]. The benchmark of dynamic regrets is more stringent than that of static ones. Using dynamic regrets, the online optimization problem becomes insolvable in the worst case where obtaining sublinear regrets could be impossible. Usually, the difficulty is characterized via a complexity measure that captures the variation in the minimizer sequence [17], [19], [20]. In [17], a dynamic mirror descent algorithm is developed for online programming, where a time-varying sequence following a given dynamics is involved. In [18], the dynamic stochastic optimization problem is investigated, where a complexity measure based on the variations in the cost function is introduced. In [19], an adaptive algorithm is designed for online optimization, where the regret bound is expressed in terms of the variations of both the cost function and the minimizer sequence. In [20], an advanced online distributed strategy is developed based on the consensus algorithm and the mirror descent algorithm, where the communication graph is modeled as a fixed undirected graph and agents track the global minimizer while exchanging local information. Nevertheless, all the aforementioned investigations assume that the cost functions allocated to each agent are convex.

Similar to convex optimization, optimization with strongly pseudoconvex cost functions, sometimes called strongly pseudoconvex optimization [21], is also a significant problem that remains to be dealt with. Strongly pseudoconvex optimization may be a nonconvex optimization problem. It appears in widespread applications such as fractional programming [25], economics [26], and frictionless contact analysis [27]. Inspired by [17]–[20], we try to study the optimization problem with strongly pseudoconvex cost functions in an online and distributed manner.

In this paper, we present an auxiliary optimization-based online distributed algorithm for online distributed strongly pseudoconvex optimization via a network of agents. Under the proposed algorithm, each agent adjusts its state value by solving an auxiliary optimization problem involving its own cost function information as well as the local states information. We employ the dynamic regret to measure performance of the algorithm. Different from online convex optimization [17]–[20], due to pseudoconvexity of cost functions, the basic convex inequality associated with gradients of cost functions, which is necessary in the analysis of regret bound in [17]–[20], does not hold any more. This brings challenges to establishing the bound of the dynamic regret. We overcome this by using Lipschitz continuity of cost functions and strong-pseudomonotonicity of cost functions' gradients. Moreover, compared with [20], a weaker condition associated with connectivity of the communication topology is used. We model the underlying communication topology as a time-varying directed graph

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sequence. We prove that if the graph sequence is B-strongly connected, then each dynamic regret function is bounded by the product of a term depending on the deviation of the minimizer sequence and a sublinear function of the learning time.

This paper is organized as follows. In Section II, mathematical preliminaries on pseudoconvex analysis and graph theory are introduced. In Section III, we formulate the problem and present the auxiliary optimization-based online distributed algorithm. In Section IV, we state our main result and give its proof. In Section V, simulation examples are presented. Section VI concludes the whole paper.

Notation: We use $|a|$ to represent the absolute value of scalar a. $\mathbb{R}$ denotes the set of real numbers. $\mathbb{N}$ is used to represent the set of positive integers. For any $T \in \mathbb{N}$, we denote set $\lfloor T \rfloor = \{0, 1, \ldots, T\}$. Let $\mathbb{R}^m$ be the m-dimensional real vector space. For given vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ and matrix $\mathbf{P} \in \mathbb{R}^{m \times m}$, we denote $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{P}} = \langle \mathbf{P}\mathbf{x}, \mathbf{y} \rangle$, $\|\mathbf{x}\|_{\mathbf{P}}^2 = \mathbf{x}^T \mathbf{P} \mathbf{x}$, $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T \mathbf{x}}$, and $\|\mathbf{x}\|_1 = \sum_{j=1}^m |\mathbf{x}_i|$, where $\mathbf{x}_i$ represents the ith entry of vector $\mathbf{x}$. For differentiable function $f(\cdot) : \mathbb{R}^m \to \mathbb{R}$, we denote the gradient of $f(\mathbf{x})$ with respect to $\mathbf{x}$ by $\nabla f(\mathbf{x})$, and use $\nabla^2 f(\mathbf{x})$ to denote its Hessian matrix. We use $\mathbf{I}_n$ to denote an $n \times n$ identity matrix. $\mathbf{1} \in \mathbb{R}^m$ denotes the m-dimensional vector with elements being all ones. For a matrix $\mathbf{A}$, $[\mathbf{A}]_{ij}$ denotes the matrix entry in the ith row and jth column, $[\mathbf{A}]_{i\cdot}$ represents the ith row of the matrix $\mathbf{A}$. $\lambda_{\max}(\mathbf{A})$ and $\lambda_{\min}(\mathbf{A})$ represent the maximal eigenvalue and the minimal eigenvalue of $\mathbf{A}$, respectively. We denote $\|\mathbf{A}\| = \sqrt{\lambda_{\max}(\mathbf{A}^T \mathbf{A})}$.

II. Preliminaries

A. Pseudoconvex Analysis

Let us begin with introducing some definitions of pseudoconvex functions and pseudomonotone mappings, which can be found in [28].

Definition 1: A differentiable function $f : \mathbb{R}^m \to \mathbb{R}$ is pseudoconvex on $\Omega \subset \mathbb{R}^m$ if for every pair of distinct points $\mathbf{x}, \mathbf{y} \in \Omega$, $\langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \geq 0$ implies $f(\mathbf{y}) - f(\mathbf{x}) \geq 0$. Moreover, f is β-strongly pseudoconvex on Ω if for every pair of distinct points $\mathbf{x}, \mathbf{y} \in \Omega$, $\langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \geq 0$ implies $f(\mathbf{y}) - f(\mathbf{x}) \geq \beta/2\|\mathbf{y} - \mathbf{x}\|^2$ with some constant $\beta > 0$.

A well-known Karush–Kuhn–Tucker (KKT) condition, which reveals the relationship between pseudoconvex optimization and variational inequality, is introduced in the following lemma.

Lemma 1 (see [29]). Suppose $f : \mathbb{R}^m \to \mathbb{R}$ is differentiable and pseudoconvex on convex set $\Omega \subset \mathbb{R}^m$. Then, $\mathbf{x}^*$ is a minimum point of f on Ω if and only if the following variational inequality holds:

$$\langle \nabla f(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^* \rangle \geq 0 \quad \forall \mathbf{x} \in \Omega.$$

Definition 2: A mapping $\mathbf{G} : \mathbb{R}^m \to \mathbb{R}^m$ is pseudomonotone on $\Omega \subset \mathbb{R}^m$ if for every pair of distinct points $\mathbf{x}, \mathbf{y} \in \Omega$, $\langle \mathbf{G}(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \geq 0$ implies $\langle \mathbf{G}(\mathbf{y}), \mathbf{y} - \mathbf{x} \rangle \geq 0$. Moreover, $\mathbf{G}$ is θ-strongly pseudomonotone on Ω if for each pair of distinct points $\mathbf{x}, \mathbf{y} \in \Omega$, $\langle \mathbf{G}(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \geq 0$ implies $\langle \mathbf{G}(\mathbf{y}), \mathbf{y} - \mathbf{x} \rangle \geq \theta\|\mathbf{y} - \mathbf{x}\|^2$ with some constant $\theta > 0$.

Clearly, from Definitions 1 and 2, convexity (respectively, strong convexity) implies pseudoconvexity (respectively, strong pseudoconvexity), and monotonicity (respectively, strong monotonicity) implies pseudomonotonicity (respectively, strong pseudomonotonicity), but not vice versa. Actually, a pseudoconvex function may be nonconvex.

B. Basic Graph Theory

The communication graph is denoted by a time-varying directed graph sequence $\{\mathcal{G}(t)\}$, $t = 0, 1, \ldots$, where $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathbf{A}(t))$, $\mathcal{V} = \{1, \ldots, n\}$ is a set of vertices, $\mathcal{E}(t) \subset \mathcal{V} \times \mathcal{V}$ is an edge set, and the weighted matrix $\mathbf{A}(t) = (a_{ij}(t))_{n \times n}$ is a nonnegative matrix for adjacency weights of edges such that $a_{ij}(t) > \ell$ for some $\ell > 0$ if $(j, i) \in \mathcal{E}(t)$ and $a_{ij}(t) = 0$ otherwise. Denote $\mathcal{N}_i(t) = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}(t)\}$ to represent the neighbor set at time t. Here, we assume $i \in \mathcal{N}_i(t)$ for all $i \in \mathcal{V}$ and $t \geq 0$. For a fixed topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{A})$, a path of length r from node i_1 to node i_{r+1} is a sequence of $r + 1$ distinct nodes $i_1 \ldots, i_{r+1}$ such that $(i_q, i_{q+1}) \in \mathcal{E}$ for $q = 1, \ldots, r$. If there exists a path between any two nodes, then $\{\mathcal{G}(t)\}$ is said to be strongly connected. For $\{\mathcal{G}(t)\}$, an B-edge set is defined as $\mathcal{E}_B(t) = \cup_{k=tB}^{(t+1)B-1} \mathcal{E}(k)$ for some constant $B > 0$. We call that $\{\mathcal{G}(t)\}$ is B-strongly connected if the directed graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}_B(t)$ is strongly connected for any $t \geq 0$.

Here, we make the following assumptions for the communication graph.

Assumption 1: $\mathbf{A}(t)$ is a doubly stochastic matrix for any $t \geq 0$, which implies that $\mathcal{G}(t)$ is balanced for any $t \geq 0$.

Assumption 2: $\{\mathcal{G}(t)\}$ is B-strongly connected.

In distributed coordination control, the weak ergodicity of stochastic matrix chains $\{\mathbf{A}(t)\}$ plays an important role in rendering agents to reach a common state [1], [2], [22]–[24]. For any $t \geq s$, we denote

$$\begin{cases} \Phi(t, s) = \mathbf{A}(t-1) \cdots \mathbf{A}(s+1)\mathbf{A}(s), \text{ if } t > s; \\ \Phi(t, s) = \mathbf{I}_n, \text{ if } t = s. \end{cases} \tag{1}$$

Based on [2, Proposition 1], under Assumptions 1 and 2, for any $i, j \in \mathcal{V}$ and $t \geq s$, we have

$$\left| [\Phi(t, s)]_{ij} - \frac{1}{n} \right| \leq H\lambda^{t-s} \tag{2}$$

where $H = 2(1 + \ell^{-(n-1)B})/(1 + \ell^{(n-1)B})$ and $\lambda = (1 - \ell^{(n-1)B})^{1/((n-1)B)}$.

Note $0 < \lambda < 1$, by (2), we know that the product of stochastic matrices $\mathbf{A}(t)$ exponentially converges to a rank-one matrix $\frac{1}{n}\mathbf{1}\mathbf{1}^T$.

III. Problem Formulation

In this section, we will formulate the problem to be studied, and present an auxiliary optimization-based online distributed algorithm.

A. Online Distributed Optimization

Let us describe a scenario of online distributed optimization. Consider a multiagent system consisting of n agents, labeled by set $\mathcal{V} = \{1, \ldots, n\}$. Agents communicate with each other via a time-varying directed graph sequence $\{\mathcal{G}(t)\}$. For agent $i \in \mathcal{V}$, a set of cost functions are given by $\{f_i^1, \ldots, f_i^T\}$, where $f_i^t : \mathbf{X} \to \mathbb{R}$ is twice differentiable for any $t \in \lfloor T \rfloor$, $T \in \mathbb{N}$ is unknown to the agents, and $\mathbf{X} \subset \mathbb{R}^m$. At each iteration time $t \in \lfloor T \rfloor$, agent i selects a state $\mathbf{x}_i(t) \in \mathbf{X}$. After the state is selected, a local cost function f_i^t is received by agent i, that is, information on cost functions is not available before decisions are made by agents. In this scenario, at each iteration time t, the goal of agents is to cooperatively solve the following optimization problem:

$$\min f^t(\mathbf{x}) = \sum_{i=1}^n f_i^t(\mathbf{x}), \text{ subject to } \mathbf{x} \in \mathbf{X}. \tag{3}$$

An online algorithm to optimize (3) should mimic the performance of its offline counterpart, and the gap between them is called the regret. If the offline problem is to minimize $\sum_{t=0}^T f^t(\mathbf{x})$, then the regret is called a static regret [13], which can be defined as

$$\mathcal{R}_i^s(T) = \sum_{t=0}^T f^t(\mathbf{x}_i(t)) - \sum_{t=0}^T f^t(\mathbf{x}^*), \quad i \in \mathcal{V} \tag{4}$$

where $\mathbf{x}^* = \mathrm{argmin}_{\mathbf{x}\in\mathbf{X}} \sum_{t=0}^{T} f^t(\mathbf{x})$. An online algorithm for (3) performs well if its regret (4) is sublinear with respect to T, i.e., $\lim_{T\to\infty} \mathcal{R}^s(T)/T = 0$ for any $i \in \mathcal{V}$. If the offline problem is to minimize $f^t(\mathbf{x})$ at each time, the regret is called a dynamic regret [16], which can be defined as

$$\mathcal{R}_i^d(T) = \sum_{t=0}^{T} f^t(\mathbf{x}_i(t)) - \sum_{t=0}^{T} f^t(\mathbf{x}^*(t)), \quad i \in \mathcal{V} \tag{5}$$

where $\mathbf{x}^*(t) = \mathrm{argmin}_{\mathbf{x}\in\mathbf{X}} f^t(\mathbf{x})$ for any $t \in \lfloor T \rfloor$. In this paper, we use dynamic regret (5), whose benchmark is more stringent than that of static ones. It is well known that using dynamic regret causes the problem insolvable in the worst case. Motivated by [17]–[20], we use the following deviation of the minimizer sequence $\{\mathbf{x}^*(t)\}_{t=0}^{T}$ to describe the difficulty:

$$\Theta_T = \sum_{t=0}^{T} \|\mathbf{x}^*(t+1) - \mathbf{x}^*(t)\|. \tag{6}$$

Throughout this paper, the following assumption is made.

Assumption 3: $\mathbf{X}$ is nonempty and bounded, i.e., there exists a positive κ such that $\|\mathbf{x} - \mathbf{y}\| \le \kappa$ for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. Moreover, $\mathbf{X}$ is closed and convex.

From Assumption 3, we know that $\mathbf{X}$ is compact. In [15] and [16], cost functions are assumed to be convex. Different from them, we make the following assumption on cost functions.

Assumption 4: For any $t \in \lfloor T \rfloor$, each f^t is μ-strongly pseudoconvex on $\mathbf{X}$.

Assumption 4 implies that ∇f^t is μ-strongly pseudomonotone on $\mathbf{X}$ since it is twice differentiable [28]. Suppose that agent i can communicate with its neighbors via the communication graph sequence $\{\mathcal{G}(t)\}$, and has access to the information associated with $f_i^t(\mathbf{x})$ after the decision is made for any $i \in \mathcal{V}$ and $t \ge 0$. The goal of this paper is to design an online distributed strategy for agents to solve (3), the dynamic regret (5) is used to measure the performance of our algorithm.

B. Auxiliary Optimization-Based Online Distributed Algorithms

In this section, an online distributed algorithm to solve (3) will be presented based on an auxiliary optimization problem. Let us begin with considering an offline and centralized optimization problem

$$\min f(\mathbf{x}), \text{ subject to } \mathbf{x} \in \mathbf{X} \tag{7}$$

where $f : \mathbb{R}^m \to \mathbb{R}$ is a strongly pseudoconvex function, and $\mathbf{X} \subset \mathbb{R}^m$ is a convex set. By Lemma 1, we know that solving (7) is equivalent to finding a $\mathbf{x}$ in $\mathbf{X}$ such that

$$\langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \ge 0 \quad \forall\, \mathbf{y} \in \mathbf{X}. \tag{8}$$

Now construct an auxiliary problem as follows:

$$\min \mathbf{x}^T \mathbf{P} \mathbf{x} + \langle \eta \nabla f(\mathbf{x}_0) - 2\mathbf{P}\mathbf{x}_0, \mathbf{x} \rangle, \text{ subject to } \mathbf{x} \in \mathbf{X} \tag{9}$$

where $\mathbf{x}_0 \in \mathbf{X}$, $\eta > 0$, and $\mathbf{P} \in \mathbb{R}^{m\times m}$ is symmetric and positive definite. By KKT condition, $\mathbf{x}^* \in \mathbf{X}$ is the solution to (9) if and only if

$$\langle 2\mathbf{P}\mathbf{x}^* + \eta \nabla f(\mathbf{x}_0) - 2\mathbf{P}\mathbf{x}_0, \mathbf{y} - \mathbf{x}^* \rangle \ge 0 \quad \forall \mathbf{y} \in \mathbf{X}. \tag{10}$$

Comparing (8) and (10), we know that $\mathbf{x}^*$ is also the solution to (7) if $\mathbf{x}^* = \mathbf{x}_0$. Replacing $\mathbf{x}_0$ with $\mathbf{x}(t)$ and replacing $\mathbf{x}^*$ with $\mathbf{x}(t+1)$, an auxiliary optimization-based strategy can be achieved as follows:

$$\mathbf{x}(t+1) = \arg\min_{\mathbf{x}\in\mathbf{X}} \left\{ \mathbf{x}^T \mathbf{P}\mathbf{x} + \langle \eta \nabla f(\mathbf{x}(t)) - 2\mathbf{P}\mathbf{x}(t), \mathbf{x} \rangle \right\}. \tag{11}$$

Note that (7) is solved if $\mathbf{x}(t+1) = \mathbf{x}(t)$ in (11), which implies that the equilibrium point of (11) is the solution to (7). Let $\mathbf{X} = \mathbb{R}^m$ and $\mathbf{P} = \mathbf{I}_m$, if f_i^t is convex, we have $\arg\min_{\mathbf{x}\in\mathbf{X}} \{\mathbf{x}^T\mathbf{P}\mathbf{x} + \langle \eta(t)\nabla f(\mathbf{x}(t)) - 2\mathbf{P}\mathbf{x}(t), \mathbf{x}\rangle\} = \mathbf{x}(t) - \frac{\eta(t)}{2}\nabla f(\mathbf{x}(t))$. Accordingly, (11) becomes the traditional gradient descent algorithm. Thus, (11) can be viewed as an extension of gradient descent algorithm, and one can intuitively assert that it is convergent. The convergence of algorithm (11) is proved in details in [30]–[32].

To solve (3), an extension of (11) to the online and distributed setting, which is called an auxiliary optimization-based online distributed algorithm by us, is proposed for each agent

$$\begin{cases} \mathbf{x}_i(t+1) = \arg\min_{\mathbf{x}\in\mathbf{X}} \left\{ \mathbf{x}^T\mathbf{P}\mathbf{x} + \langle \eta(t)\nabla f_i^t(\mathbf{x}_i(t)) \right. \\ \qquad\qquad\qquad \left. - 2\mathbf{P}\mathbf{y}_i(t), \mathbf{x} \rangle \right\} \\ \mathbf{y}_i(t) = \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)\mathbf{x}_j(t) \end{cases} \tag{12}$$

for any $i \in \mathcal{V}$, where $\mathbf{x}_i(t)$ is agent i s state at $t \in \lfloor T \rfloor$, $\mathbf{x}_i(0) = \mathbf{x}_{i0} \in \mathbf{X}$, and $\eta(t)$ is a positive and decaying learning rate with initial value $\eta(0) = \eta_0 > 0$. Algorithm (12) is designed by combining the consensus algorithm and auxiliary optimization-based strategy (11). The consensus term $\mathbf{y}_i(t)$ is motivated by the consensus algorithm in [1], [7], and [23]. Algorithm (12) runs by using the gradient information of the local cost function in the past time and the state information received from its neighbors. Therefore, algorithm (12) is online and distributed.

Remark 1: When agents update their states by (12), a common symmetric and positive definite matrix $\mathbf{P}$ is involved, which may prevent the proposed algorithm from being fully distributed. In fact, in a balanced and periodically strongly connected communication graph, it is not difficult to determine a common constant $\mathbf{P}$ for each agent based on local information. For example, the local initial state of each agent is set to be a positive definite matrix $\mathbf{P}_i$, $i \in \mathcal{V}$. We can select a strictly diagonally dominant matrix $\mathbf{P}_i$ to ensure that it is positive definite. Using the average-consensus algorithm in [1], all agents' states will converge to the positive definite matrix $\mathbf{P} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{P}_i$. Moreover, for (12), since $\mathbf{P}$ is positive definite and $\mathbf{X}$ is nonempty, $\mathbf{x}_i(t)$ exists and is unique for any $i \in \mathcal{V}$ and $t \in \lfloor T \rfloor$. Particularly, let $\mathbf{X} = \mathbb{R}^m$ and $\mathbf{P} = \mathbf{I}_m$, if f_i^t is convex, we have $\arg\min_{\mathbf{x}\in\mathbf{X}} \{\mathbf{x}^T\mathbf{P}\mathbf{x} + \langle \eta(t)\nabla f_i^t(\mathbf{x}_i(t)) - 2\mathbf{P}\mathbf{y}_i(t), \mathbf{x}\rangle\} = \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)\mathbf{x}_j(t) - \frac{\eta(t)}{2}\nabla f_i^t(\mathbf{x}_i(t))$. Then, (12) is reduced to the "consensus+gradient" algorithm [2], [11]. That is, (12) can be viewed as an extension of the "consensus+gradient" algorithm. Different from them, here, we are committed to investigating the case where the sum of cost functions is strongly pseudoconvex, rather than convex.

IV. Main Result

In this section, we will state our main result and give its proof in details. Let us begin with presenting the main result.

Theorem 1: Under Assumptions 1–4, if the learning rate in algorithm (12) is given by $\eta(t) = \alpha/\sqrt{t+1}$ for some $\alpha > 0$, then for any $i \in \mathcal{V}$ and learning time $T \in \mathbb{N}$,

$$\mathcal{R}_i^d(T) \le n\delta\sqrt{\mathcal{Q} + \frac{16\hbar L\Theta_T}{\alpha\mu\ln 2}}\left((T+1)^{3/4}\sqrt{\ln(T+1)}\right) \tag{13}$$

where $\mathcal{Q} = \frac{(4\rho_1/(\alpha\mu)+2\mathcal{K}_1)+3\alpha^2(4\rho_2/(\alpha\mu)+2\mathcal{K}_2)}{\lambda(1-\lambda)\ln 2} + \frac{6\alpha n\delta^2}{\mu\theta} + \frac{4d}{\alpha\mu\ln 2}$, $\mathcal{K}_1 = \mathcal{C}^2 + \frac{\mathcal{C}\delta H n\sqrt{m}\eta_0}{\theta(1-\lambda)}$, $\mathcal{K}_2 = \frac{m(n\delta H)^2}{4\theta^2(1-\lambda)}$, $\mathcal{C} = H\sqrt{m}\sum_{i=1}^{n}\|\mathbf{x}_i(0)\|_1$, $\rho_1 = n(5L\mathcal{K}_1 + (\kappa\sigma+\delta)\alpha\mathcal{C})$, $\rho_2 = \frac{n(5L\theta\mathcal{K}_2 + \delta H n\sqrt{m}(\kappa\sigma+\delta))}{\theta}$, $d = nL\kappa^2$, $\hbar = \sup_{\mathbf{x}\in\mathbf{X}}\|\mathbf{x}\|$, $\delta = \sup_{t\in\lfloor T\rfloor, i\in\mathcal{V}, \mathbf{x}\in\mathbf{X}}\|\nabla f_i^t(\mathbf{x})\|$, $\sigma = \sup_{t\in\lfloor T\rfloor, i\in\mathcal{V}, \mathbf{x}\in\mathbf{X}}\|\nabla^2 f_i^t(\mathbf{x})\|$, $L = \lambda_{\max}(\mathbf{P})$, $\theta = \lambda_{\min}(\mathbf{P})$, H and λ are defined in (2), Θ_T is defined in (6), and m is the dimension of the decision variable.

In Theorem 1, the bound of the dynamic regret measures the performance of the proposed online distributed algorithm. From (13), we know that Θ_T is a significant factor that influences the sublinearty of the bound. Note that $(T+1)^{3/4}\sqrt{\ln(T+1)}$ sublinearly grows with T, i.e., $\lim_{T\to\infty}\frac{(T+1)^{3/4}\sqrt{\ln(T+1)}}{T}=0$. If the global cost function f^t varies extremely slowly and Θ_T is upper bounded and small, then (3) is approximately reduced to a time-invariant case. In fact, for (13), to guarantee that the bound of the dynamic regret sublinearly grows with learning time T, it is unnecessary for Θ_T to be bounded. If Θ_T sublinearly grows with $\frac{\sqrt{T+1}}{\ln(T+1)}$, then, $\lim_{T\to\infty}\frac{\sqrt{\Theta_T}(T+1)^{3/4}\sqrt{\ln(T+1)}}{T}=0$. In this case, the gap between the online algorithm and its offline counterpart asymptotically converges to zero, and then the online distributed algorithm (12) performs well. Thus, algorithm (12) benefits for solving fractional programming problems and frictionless contact analysis in dynamic environments. If minimizer sequence $\{\mathbf{x}^*(t)\}_{t=0}^T$ fluctuates drastically, Θ_T could become linear with $\frac{\sqrt{T+1}}{\ln(T+1)}$, then, the bound in Theorem 1 cannot keep the dynamic regret sublinear. This is natural since even in the online convex optimization [17]–[20], the problem is insolvable in worst cases.

In addition, the term in the right-hand side of (13) is also influenced by connectivity of the network. Note that the lower bound ℓ of weights is not larger than $1/2$ since each agent is its own neighbor, then $\lambda\geq 1/2$ for any $n\geq 2$. By definitions of λ and H in (2), it is not difficult to verify that when $\lambda\geq 1/2$, all values of λ, $\frac{1}{1-\lambda}$, $\frac{1}{\lambda(1-\lambda)}$, as well as H, increase as the connected period B increases. Then, by the expression of $\mathcal{Q}$, we know that $\mathcal{Q}$ increases if B increases. Consequently, a larger connected period enlarges the bound of the dynamic regret.

Before giving the proof of Theorem 1, we present some useful lemmas. First, an upper bound of the error between each agent's state and their average value at each iteration time under (12) is presented.

Lemma 2: Under Assumptions 1–3, for any $t\in\lfloor T\rfloor$ and $i\in\mathcal{V}$, we have

$$\|\mathbf{x}_i(t)-\bar{\mathbf{x}}(t)\|\leq\mathcal{C}\lambda^t+\frac{\delta H\sqrt{m}}{\theta}\sum_{\tau=0}^{t}\lambda^{t-\tau}\eta(\tau) \tag{14}$$

$$\text{and}\ \ \|\mathbf{x}_i(t)-\bar{\mathbf{x}}(t)\|^2\leq\mathcal{K}_1\lambda^t+\mathcal{K}_2\sum_{\tau=0}^{t}\lambda^{t-\tau}(\eta(\tau))^2 \tag{15}$$

where $\bar{\mathbf{x}}(t)=\frac{1}{n}\sum_{i=1}^n\mathbf{x}_i(t)$.

Proof: See Appendix A. ■

Inequality (14) reflects the state error among agents in the network during evolution of the system. It is obvious that H and λ are significant factors that influence the error bound. If the connected period B increases, then H and λ increase. Accordingly, the error bound also increases. This is consistent with influence of the connected period on the bound of regret in Theorem 1. Particularly, λ acts as same as the second largest eigenvalue of the weighted matrix corresponding to a fixed and connected undirected graph [20]. Next, we will present a bound on the accumulated square error between the average of agents' states and the minimizer.

Lemma 3: Under Assumptions 1–4, if $\eta(t)$ is nonincreasing, then

$$\begin{aligned}&\sum_{t=0}^{T}\|\bar{\mathbf{x}}(t)-\mathbf{x}^*(t)\|^2\\&\leq\frac{2\rho_2}{\mu\eta(T)}\sum_{t=0}^{T}\sum_{\tau=0}^{t+1}\lambda^{t-\tau}(\eta(\tau))^2+\frac{2\rho_1}{\mu\eta(T)}\sum_{t=0}^{T}\lambda^t\\&\quad+\frac{1}{\mu\eta(T)}\sum_{t=0}^{T}\frac{n(\delta\eta(t))^2}{\theta}+\frac{8\hbar L\Theta_T}{\mu\eta(T)}+\frac{2d}{\mu\eta(T)}.\end{aligned} \tag{16}$$

Proof: See Appendix B. ■

Inequality (16) gives the bound of the average tracking error during evolution of the system. From (16), we see that deviation Θ_T plays an important role in controlling the bound of the tracking error. On the basis of Lemmas 2 and 3, we can obtain the bound of $\sum_{t=0}^T\|\mathbf{x}_i(t)-\mathbf{x}^*(t)\|$, which is used in the proof of Theorem 1. See the following proof.

Proof of Theorem 1: From Lemmas 2 and 3, for any $i\in\mathcal{V}$, we have

$$\begin{aligned}&\sum_{t=0}^{T}\|\mathbf{x}_i(t)-\mathbf{x}^*(t)\|^2\\&\leq 2\sum_{t=0}^{T}\|\bar{\mathbf{x}}(t)-\mathbf{x}^*(t)\|^2+2\sum_{t=0}^{T}\|\mathbf{x}_i(t)-\bar{\mathbf{x}}(t)\|^2\\&\leq\left(\frac{4\rho_1}{\mu\eta(T)}+2\mathcal{K}_1\right)\sum_{t=0}^{T}\lambda^t+\frac{4d}{\mu\eta(T)}\\&\quad+\left(\frac{4\rho_2}{\mu\eta(T)}+2\mathcal{K}_2\right)\sum_{t=0}^{T}\sum_{\tau=0}^{t+1}\lambda^{t-\tau}(\eta(\tau))^2\\&\quad+\frac{2}{\mu\eta(T)}\sum_{t=0}^{T}\frac{n(\delta\eta(t))^2}{\theta}+\frac{16\hbar L\Theta_T}{\mu\eta(T)}.\end{aligned} \tag{17}$$

Note that $\sum_{t=0}^T\frac{1}{t+1}=\sum_{t=1}^{T+1}\frac{1}{t}\leq 1+\int_1^{T+1}\frac{1}{t}dt=1+\ln(T+1)$ and $\sum_{t=0}^T\lambda^t\leq\frac{1}{1-\lambda}$, let the learning rate $\eta(t)$ equal to $\alpha/\sqrt{t+1}$, we have

$$\begin{aligned}\sum_{t=0}^{T}\sum_{\tau=0}^{t+1}\lambda^{t-\tau}(\eta(\tau))^2&=\sum_{\tau=1}^{T+1}\sum_{t=\tau-1}^{T}\lambda^{t-\tau}(\eta(\tau))^2+\sum_{t=0}^{T}\lambda^t(\eta_0)^2\\&\leq\frac{\alpha^2(1+\ln(T+1)+\lambda)}{\lambda(1-\lambda)}\\&\leq\frac{3\alpha^2\ln(T+1)}{\lambda(1-\lambda)\ln 2}\end{aligned} \tag{18}$$

where the first equation holds by changing the order of summations and the last inequality results from the fact $\ln(T+1)\geq\ln 2$ for any $T\geq 1$. Using Jensen's inequality yields

$$\left(\sum_{t=0}^{T}\|\mathbf{x}_i(t)-\mathbf{x}^*(t)\|\right)^2\leq(T+1)\sum_{t=0}^{T}\|\mathbf{x}_i(t)-\mathbf{x}^*(t)\|^2. \tag{19}$$

By inequalities (17)–(19), we have

$$\sum_{t=0}^{T}\|\mathbf{x}_i(t)-\mathbf{x}^*(t)\|\leq\sqrt{\mathcal{Q}+\frac{16\hbar L\Theta_T}{\alpha\mu\ln 2}}\left((T+1)^{3/4}\sqrt{\ln(T+1)}\right) \tag{20}$$

for any $i\in\mathcal{V}$. Note that $\|\nabla f_i^t(\mathbf{x})\|\leq\delta$ for any $t\in\lfloor T\rfloor$ and $i\in\mathcal{V}$ if $\mathbf{x}\in\mathbf{X}$, it implies f_i^t is δ-Lipschitz continuous on $\mathbf{X}$. We have

$$\begin{aligned}\mathcal{R}_i^d(T)&=\sum_{t=0}^{T}(f^t(\mathbf{x}_i(t))-f^t(\mathbf{x}^*(t)))\\&\leq\sum_{t=0}^{T}\sum_{j=1}^{n}\left\|f_j^t(\mathbf{x}_i(t))-f_j^t(\mathbf{x}^*(t))\right\|\\&\leq n\delta\sum_{t=0}^{T}\|\mathbf{x}_i(t)-\mathbf{x}^*(t)\|\end{aligned} \tag{21}$$

for any $i\in\mathcal{V}$. Submitting (20) into (21) yields (13). This leads to the validity of the result. ■

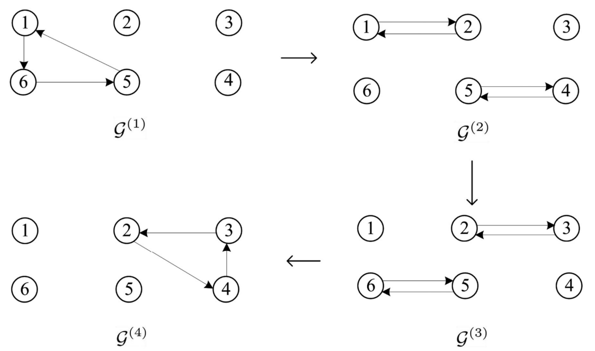

Fig. 1. Time-varying directed graph sequence.

Remark 2: If the cost function f^t in (3) is convex, the basic convex inequality $f^t(\mathbf{x}_i(t)) - f^t(\mathbf{x}^*(t)) \leq \langle \nabla f^t(\mathbf{x}_i(t)), \mathbf{x}_i(t) - \mathbf{x}^*(t)\rangle$ holds, which plays an important role in analyzing the bound of regrets [11]–[20]. The difficulties in the study of online distributed strongly pseudoconvex optimization come from the difference between inequality (21) and this basic convex inequality. To achieve the upper bound of regrets, Jensen's inequality in (19) is used to estimate the bound of $\sum_{t=0}^{T} \|\mathbf{x}_i(t) - \mathbf{x}^*(t)\|$ based on the bound of $\sum_{t=0}^{T} \|\mathbf{x}_i(t) - \mathbf{x}^*(t)\|^2$, which amplifies the bound by $\sqrt{T+1}$ times. This difficulty is yet to be overcome, and we will try to solve it in future works.

V. Simulations

In this section, we give a numerical example to illustrate the obtained result. Consider a multiagent system with six agents, labeled by index set $\{1, \ldots, 6\}$, where each agent's state is defined as $\mathbf{x}_i = [x_{i1}, x_{i2}]^T \in \mathbb{R}^2$. The agents communicate with each other via a time-varying directed graph sequence given in Fig. 1, where $\mathcal{G}^{(k)}$, $k = 1, \ldots, 4$ are four possible graphs. The switching order is given by $\mathcal{G}^{(1)} \to \mathcal{G}^{(2)} \to \mathcal{G}^{(3)} \to \mathcal{G}^{(4)} \to \mathcal{G}^{(1)} \to \cdots$. The weight of each edge is assumed to be $a_{ij} = \frac{1}{|\mathcal{N}_i(t)|}$, where $|\mathcal{N}_i(t)|$ is the number of agent i s neighbors and any agent is assumed to be its own neighbor. It is obvious that the union of the graphs is strongly connected with $B = 4$.

For any $i \in \{1, \ldots, 6\}$, the dynamic cost function is given by $f_i^t(\mathbf{x}) = \frac{i}{63}x_1^3 + \frac{i-1}{15}(x_1^2 + x_2^2) + \frac{2i+1}{6}x_1 - \frac{2(i-3)}{3}q(t)x_2$, $\mathbf{x} = [x_1, x_2]^T$, and $t = 0, 1, \ldots, T$, here we set $T = 50$. Additionally, the constraint set is given as a box set $\mathbf{X} = \{-2 \leq x_1 \leq 1, 0 \leq x_2 \leq 3\}$. The sum is computed as $f^t(\mathbf{x}) = \frac{1}{3}x_1^3 + x_1^2 + x_2^2 + 8x_1 - 2q(t)x_2$. For any $t \geq 0$, it is obvious that f^t is not convex on $\mathbf{X}$. However, it is not difficult to verify that f^t is strongly pseudoconvex on $\mathbf{X}$. We assume $q(t) = \arctan(t/20)$. In the offline setting, we use all the information to compute the minimum value $\mathbf{x}^*(t) = [-2, \arctan(t/20)]^T$. Now suppose that agent i can only have access to its local cost function f_i^t. Initial states are given as: $\mathbf{x}_1 = \mathbf{x}_5 = [-0.5, 1]^T$; $\mathbf{x}_2 = [-1, 0]^T$; $\mathbf{x}_3 = [1, 2]^T$; $\mathbf{x}_4 = [-2, 3]^T$; and $\mathbf{x}_6 = [1, 2]^T$. Algorithm (12) is applied to the problem with $\eta(t) = 6/\sqrt{t+1}$ and $\mathbf{P} = \mathrm{diag}\{6, 30\}$. Fig. 2 shows the trajectories of agents' states, from which we can see that the state of each agent approximates to $[-2, 1.2]^T$ at $t = 50$. The average regrets are shown in Fig. 3, from which we can see that each average regret approaches to zero after a period of time. These observations are consistent with the results established in Theorem 1. Additionally, an advanced online distributed algorithm in [20] is employed to address the same strongly pseudoconvex optimization. The regret bound for convex optimization is given in [20, Corollary 4]. Under same initial conditions, the trajectories of agents' states and the average regrets are shown in Fig. 4 and Fig. 5, respectively. It can be seen that the problem cannot be solved by their method. Thus, the effectiveness of our strategy is further verified.

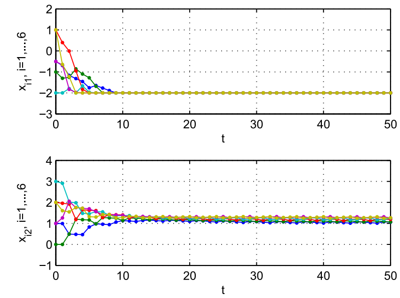

Fig. 2. Trajectories of $\mathbf{x}_i(t)$, $i = 1, \ldots, 6$ under algorithm (12).

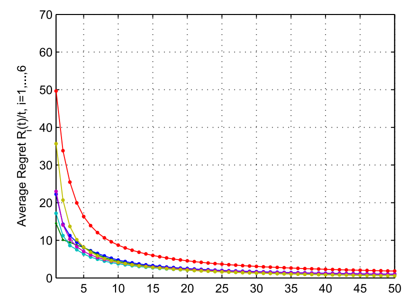

Fig. 3. Trajectories of average regrets $\mathcal{R}_i(t)/t$, $i = 1, \ldots, 6$ under algorithm (12).

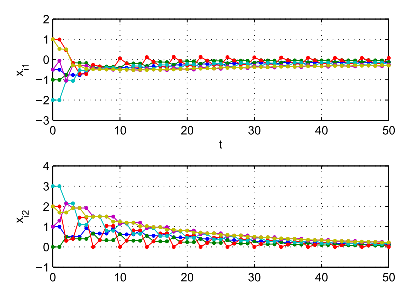

Fig. 4. Trajectories of $\mathbf{x}_i(t)$, $i = 1, \ldots, 6$ under the strategy in [20].

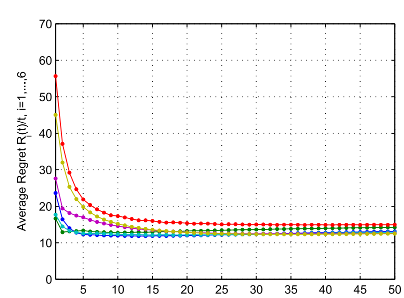

Fig. 5. Trajectories of average regrets $\mathcal{R}_i(t)/t$, $i = 1, \ldots, 6$ under the strategy in [20].

VI. Conclusion

In this paper, the problem of online distributed optimization with strongly pseudoconvex-sum cost functions has been investigated. To address this problem, we have presented an auxiliary optimization-based online distributed algorithm. By implementing the algorithm, every agent adjusts its state value by solving an auxiliary optimization problem involving its own cost function information and the local states information received from its immediate neighbors. The result shows that if the time-varying graph sequence is B-strongly connected, then each dynamic regret function is bounded by the product of a term depending on the deviation of the minimizer sequence and a sublinear function of the learning time.

How to achieve a lower bound of the regret functions is a difficult problem in online distributed strongly pseudoconvex optimization. How to solve more general nonconvex optimization in online and distributed manner is another interesting and challenging problem. These two topics will be considered in our future work. Some other issues may also be considered, such as the case with network-induced time-delays, packet loss and communication bandwidth constraints, which will bring new challenges in online distributed optimization with strongly pseudoconvex-sum cost functions.

Appendix

A. Proof of Lemma 2

Note that $\mathbf{x}_i(t+1)$ generated by algorithm (12) is the solution to the following optimization:

$$\min_{\mathbf{x}\in\mathbf{X}} \mathbf{x}^T\mathbf{P}\mathbf{x} + \left\langle \eta(t)\nabla f_i^t(\mathbf{x}_i(t)) - 2\mathbf{P}\mathbf{y}_i(t), \mathbf{x}\right\rangle.$$

By KKT condition, we have

$$\begin{aligned}&\langle \mathbf{x}_i(t+1) - \mathbf{y}_i(t), \mathbf{x}_i(t+1) - \mathbf{x}\rangle_{\mathbf{P}}\\ &\quad\le \frac{\eta(t)}{2}\left\langle \nabla f_i^t(\mathbf{x}_i(t), \mathbf{x}_i(t+1) - \mathbf{x}\right\rangle\end{aligned}$$

for any $\mathbf{x}\in\mathbf{X}$. Note that for any $\mathbf{x}_i(t)\in\mathbf{X}$ and $i\in\mathcal{V}$, by the convexity of $\mathbf{X}$, we know $\mathbf{y}_i(t)\in\mathbf{X}$. Let $\mathbf{x}=\mathbf{y}_i(t)$, due to the facts $2\theta\|\mathbf{x}_i(t+1)-\mathbf{y}_i(t)\|^2 \le \|\mathbf{x}_i(t+1)-\mathbf{y}_i(t)\|_{\mathbf{P}}^2$ and $\|\nabla f_i^t(\mathbf{x}_i(t))\|\le\delta$, it follows that

$$2\theta\|\mathbf{x}_i(t+1) - \mathbf{y}_i(t)\|^2 \le \frac{\delta\eta(t)}{2}\|\mathbf{x}_i(t+1) - \mathbf{y}_i(t)\|. \tag{22}$$

Now we denote $\mathbf{e}_i(t) = \mathbf{x}_i(t+1) - \mathbf{y}_i(t)$, $i\in\mathcal{V}$, from (22), it yields that $\|\mathbf{e}_i(t)\| \le \delta\eta(t)/(2\theta)$. Moreover,

$$\mathbf{x}_i(t+1) = \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)\mathbf{x}_j(t) + \mathbf{e}_i(t).$$

We define vector $\widetilde{\mathbf{x}}_r(t)\in\mathbb{R}^n$, which stacks up the rth entry of $\mathbf{x}_i(t)$, $i\in\mathcal{V}$. Similarly, we also define vector $\widetilde{\mathbf{e}}_r(t)\in\mathbb{R}^n$, which stacks up the rth entry of $\mathbf{e}_i(t)$, $i\in\mathcal{V}$. Then, it follows that

$$\widetilde{\mathbf{x}}_r(t+1) = \mathbf{A}(t)\widetilde{\mathbf{x}}_r(t) + \widetilde{\mathbf{e}}_r(t)$$

which implies that

$$\widetilde{\mathbf{x}}_r(t) = \Phi(t,0)\widetilde{\mathbf{x}}_r(0) + \sum_{\tau=0}^{t}\Phi(t,\tau)\widetilde{\mathbf{e}}_r(\tau) \tag{23}$$

where $\Phi(t,s)$ is defined as (1). Note that $\Phi(t,s)$ is a doubly stochastic matrix for any $t\ge s\ge 0$, then, (23) further implies that

$$\mathbf{1}^T\widetilde{\mathbf{x}}_r(t) = \mathbf{1}^T\widetilde{\mathbf{x}}_r(0) - \sum_{\tau=0}^{t}\mathbf{1}^T\widetilde{\mathbf{e}}_r(\tau). \tag{24}$$

From (23) and (24), we have

$$\begin{aligned}&\left|[\widetilde{\mathbf{x}}_r(t)]_i - \frac{1}{n}\mathbf{1}^T\widetilde{\mathbf{x}}_r(t)\right|\\ &\le\left|\left([\Phi(t,0)]_{i\cdot} - \frac{1}{n}\mathbf{1}^T\right)\widetilde{\mathbf{x}}_r(0)\right| + \sum_{\tau=0}^{t}\left|\left([\Phi(t,\tau)]_{i\cdot} - \frac{1}{n}\mathbf{1}^T\right)\widetilde{\mathbf{e}}_r(\tau)\right|\\ &\le \max_{1\le j\le n}\left|[\Phi(t,0)]_{ij} - \frac{1}{n}\right|\|\widetilde{\mathbf{x}}_r(0)\|_1\\ &\quad + \frac{n\delta}{2\theta}\sum_{\tau=0}^{t}\eta(\tau)\max_{1\le j\le n}\left|[\Phi(t,\tau)]_{ij} - \frac{1}{n}\right|\end{aligned}$$

for every $i\in\mathcal{V}$. Using (2), we have

$$\left|[\widetilde{\mathbf{x}}_r(t)]_i - \frac{1}{n}\mathbf{1}^T\widetilde{\mathbf{x}}_r(t)\right| \le H\lambda^t\|\widetilde{\mathbf{x}}_r(0)\|_1 + \frac{n\delta H}{2\theta}\sum_{\tau=0}^{t}\lambda^{t-\tau}\eta(\tau).$$

This directly implies (14). Furthermore, due to the facts that $\eta(t)$ is nonincreasing and $0<\lambda<1$, from (14), we have

$$\begin{aligned}\|\mathbf{x}_i(t) - \bar{\mathbf{x}}(t)\|^2 \le& \left(\mathcal{C}^2 + \frac{\mathcal{C}\delta Hn\sqrt{m}\eta_0}{\theta(1-\lambda)}\right)\lambda^t\\ &+ \frac{m(n\delta H)^2}{4\theta^2}\left(\sum_{\tau=0}^{t}\lambda^{t-\tau}\eta(\tau)\right)^2.\end{aligned} \tag{25}$$

Using Cauchy–Schwarz inequality yields

$$\begin{aligned}\left(\sum_{\tau=0}^{t}\lambda^{t-\tau}\eta(\tau)\right)^2 &\le \left(\sum_{\tau=0}^{t}\lambda^{t-\tau}\right)\left(\sum_{\tau=0}^{t}\lambda^{t-\tau}(\eta(\tau))^2\right)\\ &\le \frac{1}{1-\lambda}\sum_{\tau=0}^{t}\lambda^{t-\tau}(\eta(\tau))^2.\end{aligned} \tag{26}$$

Inequalities (26) and (25) lead to validity of (15). ■

B Proof of Lemma 3

To prove Lemma 3, we give the following lemma.

Lemma 4: Under Assumptions 1–3, for any $\mathbf{z}\in\mathbb{R}^m$ and $t\in\lfloor T\rfloor$,

$$\begin{aligned}&\sum_{i=1}^{n}\langle \mathbf{x}_i(t) - \mathbf{y}_i(t), \mathbf{z} - \mathbf{x}_i(t+1)\rangle_{\mathbf{P}}\\ &\le \frac{5nL}{2}\sum_{i=1}^{n}\mathcal{K}_1\lambda^t + \frac{5nL}{2}\mathcal{K}_2\sum_{\tau=0}^{t+1}\lambda^{t-\tau}(\eta(\tau))^2.\end{aligned} \tag{27}$$

Proof: Note that

$$\begin{aligned}&\sum_{i=1}^{n}\langle \mathbf{x}_i(t) - \mathbf{y}_i(t), \mathbf{z} - \mathbf{x}_i(t+1)\rangle_{\mathbf{P}}\\ &= \sum_{i=1}^{n}\langle \mathbf{x}_i(t) - \mathbf{y}_i(t), \mathbf{z} - \bar{\mathbf{x}}(t+1)\rangle_{\mathbf{P}}\\ &\quad + \sum_{i=1}^{n}\langle \mathbf{x}_i(t) - \mathbf{y}_i(t), \bar{\mathbf{x}}(t+1) - \mathbf{x}_i(t+1)\rangle_{\mathbf{P}}\\ &\le \left\langle \sum_{i=1}^{n}(\mathbf{x}_i(t) - \mathbf{y}_i(t)), \mathbf{z} - \bar{\mathbf{x}}(t+1)\right\rangle_{\mathbf{P}}\\ &\quad + L\sum_{i=1}^{n}\|\mathbf{x}_i(t) - \mathbf{y}_i(t)\|\,\|\bar{\mathbf{x}}(t+1) - \mathbf{x}_i(t+1)\|.\end{aligned}$$

It is not difficult to verify that $\sum_{i=1}^n \mathbf{P}\mathbf{y}_i(t) = \sum_{i=1}^n \mathbf{P}\mathbf{x}_i(t)$. Then, $\langle \sum_{i=1}^n (\mathbf{x}_i(t) - \mathbf{y}_i(t)), \mathbf{z} - \bar{\mathbf{x}}(t+1)\rangle_{\mathbf{P}} = 0$. Using Young's inequality and the fact that $\sum_{j=1}^n a_{ij}(\mathbf{x}_i(t) - \mathbf{x}_j(t))$ is a convex combination of the $\mathbf{x}_i(t) - \mathbf{x}_j(t)$, we have

$$\begin{aligned}\sum_{i=1}^n \langle \mathbf{x}_i(t) - \mathbf{y}_i(t), \mathbf{z} - \mathbf{x}_i(t+1)\rangle_{\mathbf{P}} &\le 2L\sum_{i=1}^n \|\mathbf{x}_i(t) - \bar{\mathbf{x}}(t)\|^2 \\ &\quad + \frac{L}{2}\sum_{i=1}^n \|\bar{\mathbf{x}}(t+1) - \mathbf{x}_i(t+1)\|^2 .\end{aligned} \tag{28}$$

Using (15) in Lemma 2 and the fact $0 < \lambda < 1$ to (28), it leads to the validity of (27). ■

Proof of Lemma 3: Consider an auxiliary function as $D(t) = \frac{1}{2}\sum_{i=1}^n \|\mathbf{x}^*(t) - \mathbf{x}_i(t)\|_{\mathbf{P}}^2$. Then, let us study the variation of D for one stage of (12)

$$\begin{aligned}\triangle D(t) &= D(t+1) - D(t) \\ &= -\frac{1}{2}\sum_{i=1}^n \|\mathbf{x}_i(t) - \mathbf{x}_i(t+1)\|_{\mathbf{P}}^2 \\ &\quad + \sum_{i=1}^n \left\langle \mathbf{x}_i(t) - \mathbf{x}_i(t+1), \mathbf{x}^*(t) - \mathbf{x}_i(t+1)\right\rangle_{\mathbf{P}} \\ &\quad + \sum_{i=1}^n \Big\langle \frac{1}{2}(\mathbf{x}^*(t+1) + \mathbf{x}^*(t)) \\ &\quad - \mathbf{x}_i(t+1), \mathbf{x}^*(t) - \mathbf{x}_i(t+1)\Big\rangle_{\mathbf{P}} \\ &\le \sum_{i=1}^n \left\langle \mathbf{x}_i(t) - \mathbf{x}_i(t+1), \mathbf{x}^*(t) - \mathbf{x}_i(t+1)\right\rangle_{\mathbf{P}} \\ &\quad - \sum_{i=1}^n \frac{\theta}{2}\|\mathbf{x}_i(t) - \mathbf{x}_i(t+1)\|^2 \\ &\quad + 2\hbar L\|\mathbf{x}^*(t+1) - \mathbf{x}^*(t)\|.\end{aligned} \tag{29}$$

Moreover, by KKT condition, we have

$$\begin{aligned}&\langle \mathbf{y}_i(t) - \mathbf{x}_i(t+1), \mathbf{x}^*(t) - \mathbf{x}_i(t+1)\rangle_{\mathbf{P}} \\ &\qquad \le \frac{\eta(t)}{2}\left\langle \nabla f_i^t(\mathbf{x}_i(t)), \mathbf{x}^*(t) - \mathbf{x}_i(t+1)\right\rangle\end{aligned}$$

for any $i \in \mathcal{V}$. Together with (27) in Lemma 4, it follows that

$$\begin{aligned}&\sum_{i=1}^n \left\langle \mathbf{x}_i(t) - \mathbf{x}_i(t+1), \mathbf{x}^*(t) - \mathbf{x}_i(t+1)\right\rangle_{\mathbf{P}} \\ &= \sum_{i=1}^n \left\langle \mathbf{y}_i(t) - \mathbf{x}_i(t+1), \mathbf{x}^*(t) - \mathbf{x}_i(t+1)\right\rangle_{\mathbf{P}} \\ &\quad + \sum_{i=1}^n \left\langle \mathbf{x}_i(t) - \mathbf{y}_i(t), \mathbf{x}^*(t) - \mathbf{x}_i(t+1)\right\rangle_{\mathbf{P}} \\ &\le \sum_{i=1}^n \frac{\eta(t)}{2}\left\langle \nabla f_i^t(\mathbf{x}_i(t)), \mathbf{x}^*(t) - \mathbf{x}_i(t+1)\right\rangle \\ &\quad + \frac{5nL}{2}\mathcal{K}_1\lambda^t + \frac{5nL}{2}\mathcal{K}_2\sum_{\tau=0}^{t+1}\lambda^{t-\tau}(\eta(\tau))^2 .\end{aligned} \tag{30}$$

From Lemma 1, one knows $\langle \sum_{i=1}^n \nabla f_i^t(\mathbf{x}^*(t)), \bar{\mathbf{x}}(t) - \mathbf{x}^*(t)\rangle \ge 0$. By Assumption 4, we know that $\sum_{i=1}^n \nabla f_i^t$ is strongly pseudomonotone, then we have

$$\left\langle \sum_{i=1}^n \nabla f_i^t(\bar{\mathbf{x}}(t)), \bar{\mathbf{x}}(t) - \mathbf{x}^*(t)\right\rangle \ge \frac{\mu}{2}\|\bar{\mathbf{x}}(t) - \mathbf{x}^*(t)\|^2.$$

Note that $\|\nabla^2 f_i^t(\mathbf{x})\| \le \sigma$ for any $\mathbf{x} \in \mathbf{X}$, it implies $\|\nabla f_i^t(\mathbf{x}_i(t)) - \nabla f_i^t(\bar{\mathbf{x}}(t))\| \le \sigma\|\mathbf{x}_i(t) - \bar{\mathbf{x}}(t)\|$ for any $i \in \mathcal{V}$. Together with boundedness of $\mathbf{X}$ in Assumption 3, we have

$$\begin{aligned}&\sum_{i=1}^n \left\langle \nabla f_i^t(\mathbf{x}_i(t)), \mathbf{x}^*(t) - \mathbf{x}_i(t)\right\rangle \\ &= \sum_{i=1}^n \left\langle \nabla f_i^t(\mathbf{x}_i(t)) - \nabla f_i^t(\bar{\mathbf{x}}(t)), \mathbf{x}^*(t) - \mathbf{x}_i(t)\right\rangle \\ &\quad + \sum_{i=1}^n \left\langle \nabla f_i^t(\bar{\mathbf{x}}(t)), \bar{\mathbf{x}}(t) - \mathbf{x}_i(t)\right\rangle - \sum_{i=1}^n \left\langle \nabla f_i^t(\bar{\mathbf{x}}(t)), \bar{\mathbf{x}}(t) - \mathbf{x}^*(t)\right\rangle \\ &\le \sum_{i=1}^n (\kappa\sigma + \delta)\|\mathbf{x}_i(t) - \bar{\mathbf{x}}(t)\| - \frac{\mu}{2}\|\bar{\mathbf{x}}(t) - \mathbf{x}^*(t)\|^2.\end{aligned}$$

Then,

$$\begin{aligned}&\eta(t)\sum_{i=1}^n \left\langle \nabla f_i^t(\mathbf{x}_i(t)), \mathbf{x}^*(t) - \mathbf{x}_i(t+1)\right\rangle \\ &= \sum_{i=1}^n \eta(t)\left\langle \nabla f_i^t(\mathbf{x}_i(t)), \mathbf{x}^*(t) - \mathbf{x}_i(t)\right\rangle \\ &\quad + \sum_{i=1}^n \eta(t)\left\langle \nabla f_i^t(\mathbf{x}_i(t)), \mathbf{x}_i(t) - \mathbf{x}_i(t+1)\right\rangle \\ &\le -\frac{\mu\eta(t)}{2}\|\bar{\mathbf{x}}(t) - \mathbf{x}^*(t)\|^2 + \sum_{i=1}^n (\kappa\sigma + \delta)\eta(t)\|\mathbf{x}_i(t) - \bar{\mathbf{x}}(t)\| \\ &\quad + \sum_{i=1}^n \delta\eta(t)\|\mathbf{x}_i(t) - \mathbf{x}_i(t+1)\| \\ &\le -\frac{\mu\eta(t)}{2}\|\bar{\mathbf{x}}(t) - \mathbf{x}^*(t)\|^2 + \sum_{i=1}^n (\kappa\sigma + \delta)\eta(t)\|\mathbf{x}_i(t) - \bar{\mathbf{x}}(t)\| \\ &\quad + \sum_{i=1}^n \frac{\theta}{2}\|\mathbf{x}_i(t) - \mathbf{x}_i(t+1)\|^2 + \frac{n(\delta\eta(t))^2}{2\theta}\end{aligned} \tag{31}$$

where the last inequality results from using Young's inequality. By (29)–(31) and using (14) in Lemma 2, we have

$$\begin{aligned}\triangle D(t) &\le -\frac{\mu\eta(t)}{4}\|\bar{\mathbf{x}}(t) - \mathbf{x}^*(t)\|^2 + \frac{n(\delta\eta(t))^2}{4\theta} \\ &\quad + \frac{\rho_1}{2}\lambda^t + \frac{\rho_2}{2}\sum_{\tau=0}^{t+1}\lambda^{t-\tau}(\eta(\tau))^2 \\ &\quad + 2\hbar L\|\mathbf{x}^*(t+1) - \mathbf{x}^*(t)\|.\end{aligned} \tag{32}$$

Due to the fact $D(t) \ge 0$ for any $t \in \lfloor T \rfloor$, there is -$\sum_{t=0}^T \triangle D(t) = D(0) - D(T) \le D(0) \le d/2$. Summing from $t = 0$ to T at both sides of (32) yields (16). ■

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